Geomagnetic Dynamo Theory

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Dieter Schmitt  
Geomagnetic Dynamo Theory
**Need of a geodynamo**

- Earth and many other celestial bodies possess a large-scale, often variable magnetic field
- Decay time $\tau = \frac{L^2}{\eta}$
  - magnetic diffusivity $\eta = \frac{c^2}{4\pi\sigma}$, electrical conductivity $\sigma$
- Earth $\tau \approx 20000$ yr
- Variability of geomagnetic field
- **Dynamo**: $\mathbf{u} \times \mathbf{B} \rightleftarrows \mathbf{j} \rightleftarrows \mathbf{B} \rightleftarrows \mathbf{u}$
  - Faraday
  - Ampere
  - Lorentz
  - motion of an electrical conductor in an ‘inducing’ magnetic field
  - induction of electric current
- **Self-excited dynamo**: inducing magnetic field created by the electric current (Siemens 1867)
Homopolar dynamo

electromotive force $\mathbf{u} \times \mathbf{B} \sim$ electric current through wire loop

$\sim$ induced magnetic field reinforces applied magnetic field

self-excitation if rotation $\Omega > 2\pi R/M$ is maintained

where $R$ resistance, $M$ inductance
Introduction
Basic electrodynamics
Kinematic, turbulent dynamos
Magnetohydrodynamical dynamos
Geodynamo simulations

Geodynamo hypothesis

- Larmor (1919): Magnetic field of Earth and Sun maintained by self-excited dynamo
- Homogeneous dynamo (no wires in Earth core)
  - complex motion necessary
- Kinematic ($u$ prescribed, linear)
- Dynamic ($u$ determined by forces, including Lorentz force, non-linear)
Pre-Maxwell theory

Maxwell equations: cgs system, vacuum, $B = H$, $D = E$

\[ c\nabla \times B = 4\pi j + \frac{\partial E}{\partial t}, \quad c\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0, \quad \nabla \cdot E = 4\pi \lambda \]

Basic assumptions of MHD:

- $u \ll c$: system stationary on light travel time, no em waves
- high electrical conductivity: $E$ determined by $\partial B/\partial t$, not by charges $\lambda$

\[ \frac{E}{L} \approx \frac{B}{T} \sim \frac{E}{B} \approx 1 \frac{L}{cT} \approx \frac{u}{c} \ll 1, \quad E \text{ plays minor role : } \frac{e_{el}}{e_m} \approx \frac{E^2}{B^2} \ll 1 \]

\[ \frac{\partial E/\partial t}{c\nabla \times B} \approx \frac{E/T}{cB/L} \approx \frac{E}{B} \frac{u}{c} \approx \frac{u^2}{c^2} \ll 1, \quad \text{displacement current negligible} \]

Pre-Maxwell equations:

\[ c\nabla \times B = 4\pi j, \quad c\nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0 \]
Pre-Maxwell theory

Pre-Maxwell equations Galilei-covariant:

\[ E' = E + \frac{1}{c} u \times B, \quad B' = B, \quad j' = j \]

Relation between \( j \) and \( E \) by Galilei-covariant Ohm’s law:

\[ j' = \sigma E' \]

in resting frame of reference, \( \sigma \) electrical conductivity

\[ j = \sigma \left( E + \frac{1}{c} u \times B \right) \]

Magnetohydrokinematics:

\[ c \nabla \times B = 4\pi j \]

\[ c \nabla \times E = -\frac{\partial B}{\partial t} \]

\[ \nabla \cdot B = 0 \]

\[ j = \sigma \left( E + \frac{1}{c} u \times B \right) \]

Magnetohydrodynamics:

additionally

Equation of motion
Equation of continuity
Equation of state
Energy equation
Evolution of magnetic field

\[
\frac{\partial B}{\partial t} = -c \nabla \times E = -c \nabla \times \left( \frac{j}{\sigma} - \frac{1}{c} u \times B \right) = -c \nabla \times \left( \frac{c}{4\pi \sigma} \nabla \times B - \frac{1}{c} u \times B \right) \\
= \nabla \times (u \times B) - \nabla \times \left( \frac{c^2}{4\pi \sigma} \nabla \times B \right) = \nabla \times (u \times B) - \eta \nabla \times \nabla \times B
\]

with \( \eta = \frac{c^2}{4\pi \sigma} = \text{const magnetic diffusivity} \)

\( \nabla \times (u \times B) = -B \nabla \cdot u + (B \cdot \nabla) u - (u \cdot \nabla) B \)

expansion/contraction, shear/stretching, advection
Alfvén theorem

Ideal conductor $\eta = 0$:

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B)$$

Magnetic flux through floating surface is constant:

$$\frac{d}{dt} \int_F B \cdot dF = 0$$

Proof:

$$0 = \int \nabla \cdot B \, dV = \int B \cdot dF = \int_F B(t) \cdot dF - \int_{F'} B(t') \cdot dF' - \oint_C B(t) \cdot ds \times u \, dt \cdot$$

$$\int_{F'} B(t + dt) \cdot dF' - \int_F B(t) \cdot dF = \int_F \{B(t + dt) - B(t)\} \cdot dF - \oint_C B \cdot ds \times u \, dt$$

$$= dt \left( \int \frac{\partial B}{\partial t} \cdot dF - \oint_C B \cdot ds \times u \right) = dt \left( \int \nabla \times (u \times B) \cdot dF - \oint_C B \cdot ds \times u \right)$$

$$= dt \left( \oint_C u \times B \cdot ds - \oint_C B \cdot ds \times u \right) = 0$$
Alfven theorem

Frozen-in field lines
impression that magnetic field follows flow, but \( \mathbf{E} = -\mathbf{u} \times \mathbf{B} \) and \( \nabla \times \mathbf{E} = -c \partial \mathbf{B} / \partial t \)

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) = -\mathbf{B} \nabla \cdot \mathbf{u} + (\mathbf{B} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{B}
\]

(i) star contraction: \( \overline{\mathbf{B}} \sim R^{-2}, \overline{\rho} \sim R^{-3} \therefore \overline{\mathbf{B}} \sim \overline{\rho}^{2/3} \)
Sun \( \sim \) white dwarf \( \sim \) neutron star: \( \rho \ [\text{g cm}^{-3}] : 1 \sim 10^6 \sim 10^{15} \)
(ii) stretching of flux tube: \( \overline{\mathbf{B}} d^2 = \text{const}, \overline{ld}^2 = \text{const} \sim \overline{B} \sim l \)
(iii) shear, differential rotation

\[ B d^2 = \text{const}, \overline{ld^2} = \text{const} \sim B \sim l \]
Differential rotation

$$\frac{\partial B_\phi}{\partial t} = r \sin \theta \nabla \Omega \cdot B_p$$
Magnetic Reynolds number

Dimensionless variables: length $L$, velocity $u_0$, time $L/u_0$

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) - R_m^{-1} \nabla \times \nabla \times B \quad \text{with} \quad R_m = \frac{u_0 L}{\eta}$$

as combined parameter

laboratorium: $R_m \ll 1$, cosmos: $R_m \gg 1$

induction for $R_m \gg 1$, diffusion for $R_m \ll 1$, e.g. for small $L$

example: flux expulsion from closed velocity fields
Flux expulsion

(Weiss 1966)
Poloidal and toroidal magnetic fields

Spherical coordinates \((r, \vartheta, \varphi)\)

**Axisymmetric fields:** \(\partial / \partial \varphi = 0\)

\[
B(r, \vartheta) = (B_r, B_\vartheta, B_\varphi)
\]

\(\nabla \cdot B = 0\)

\(B = B_p + B_t\) poloidal and toroidal magnetic field

\(B_t = (0, 0, B_\varphi)\) satisfies \(\nabla \cdot B_t = 0\)

\(B_p = (B_r, B_\vartheta, 0) = \nabla \times A\) with \(A = (0, 0, A_\varphi)\) satisfies \(\nabla \cdot B_p = 0\)

\[
B_p = \frac{1}{r \sin \vartheta} \left( \frac{\partial r \sin \vartheta A_\varphi}{r \partial \vartheta}, -\frac{\partial r \sin \vartheta A_\varphi}{\partial r}, 0 \right)
\]

axisymmetric magnetic field determined by the two scalars: \(r \sin \vartheta A_\varphi\) and \(B_\varphi\)
Poloidal and toroidal magnetic fields

**Axisymmetric fields:**

\[
\begin{align*}
\vec{j}_t &= \frac{c}{4\pi} \nabla \times \vec{B}_p, \\
\vec{j}_p &= \frac{c}{4\pi} \nabla \times \vec{B}_t
\end{align*}
\]

\( r \sin \vartheta A_\varphi = \text{const} \) : field lines of poloidal field in meridional plane

Field lines of \( \vec{B}_t \) are circles around symmetry axis

**Non-axisymmetric fields:**

\[
\begin{align*}
\vec{B} &= \vec{B}_p + \vec{B}_t = \nabla \times \nabla \times (P\vec{r}) + \nabla \times (T\vec{r}) = -\nabla \times (\vec{r} \times \nabla P) - \vec{r} \times \nabla T \\
\vec{r} &= (r, 0, 0), \quad P(r, \vartheta, \varphi) \quad \text{and} \quad T(r, \vartheta, \varphi) \quad \text{defining scalars} \\
\nabla \cdot \vec{B} &= 0, \quad \vec{j}_t = \frac{c}{4\pi} \nabla \times \vec{B}_p, \quad \vec{j}_p = \frac{c}{4\pi} \nabla \times \vec{B}_t \\
r \cdot \vec{B}_t &= 0 \quad \text{field lines of the toroidal field lie on spheres, no} \ r \ \text{component} \\
\vec{B}_p \quad \text{has in general all three components}
\end{align*}
\]
Cowling’s theorem
(Cowling 1934)

Axisymmetric magnetic fields can not be maintained by a dynamo.

Sketch of proof:
- electric currents as sources of the magnetic field only in finite space
- field line \( F = 0 \) along axis closes at infinity
- field lines on circular tori whose cross section are the lines \( F = \text{const} \)

- axisymmetry: closed neutral line
- around neutral line is \( \nabla \times \mathbf{B} \neq 0 \Leftrightarrow j_{\varphi} \neq 0 \), but there is no source of \( j_{\varphi} \):
  \( E_{\varphi} = 0 \) because of axisymmetry and \( (\mathbf{u} \times \mathbf{B})_{\varphi} = 0 \) on neutral line for finite \( \mathbf{u} \)
Consider vicinity of neutral line, assume axisymmetry
\[ \oint B_p \, dl = \oint B \cdot dl = \int \nabla \times B \, df = \frac{4\pi}{c} \int j \cdot df = \frac{4\pi}{c} \int |j_\varphi| \, df \]
\[ = \frac{4\pi \sigma}{c^2} \int |u_p \times B_p| \, df \leq \frac{4\pi \sigma}{c^2} \int u_p B_p \, df \leq \frac{4\pi \sigma}{c^2} u_{p,\text{max}} \int B_p \, df \]
integration circle of radius \( \varepsilon \)
\[ B_p 2\pi \varepsilon \leq \frac{4\pi \sigma}{c^2} u_{p,\text{max}} B_p \pi \varepsilon^2 \quad \text{or} \quad 1 \leq \frac{2\pi \sigma}{c^2} u_{p,\text{max}} \varepsilon \]
\[ \varepsilon \to 0 \quad \sim \quad u_{p,\text{max}} \to \infty \]
contradiction
**Toroidal theorems**

**Toroidal velocity theorem** (Elsasser 1947, Bullard & Gellman 1954)

A toroidal motion in a spherical conductor can not maintain a magnetic field by dynamo action.

Sketch of proof:

\[
\frac{d}{dt}(r \cdot B) = \eta \nabla^2 (r \cdot B) \quad \text{for} \quad r \cdot u = 0
\]

\[r \cdot B \rightarrow 0 \quad \text{for} \quad t \rightarrow \infty \quad P \rightarrow 0 \quad T \rightarrow 0\]

**Toroidal field theorem / Invisible dynamo theorem** (Kaiser et al. 1994)

A purely toroidal magnetic field can not be maintained by a dynamo.
Parker’s helical convection

velocity \( u \)

vorticity \( \omega = \nabla \times u \)

helicity \( H = u \cdot \omega \)
Mean-field theory

Statistical consideration of turbulent helical convection on mean magnetic field (Steenbeck, Krause and Rädler 1966)

\[
\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \eta \nabla \times \nabla \times \boldsymbol{B}
\]

\[
\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}' , \quad \boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}' \quad \text{Reynolds rules for averages}
\]

\[
\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times (\overline{\boldsymbol{u}} \times \overline{\boldsymbol{B}} + \mathcal{E}) - \eta \nabla \times \nabla \times \overline{\boldsymbol{B}}
\]

\[
\mathcal{E} = \overline{\boldsymbol{u}' \times \boldsymbol{B}'} \quad \text{mean electromotive force}
\]

\[
\frac{\partial \boldsymbol{B}'}{\partial t} = \nabla \times (\overline{\boldsymbol{u}} \times \boldsymbol{B}' + \boldsymbol{u}' \times \overline{\boldsymbol{B}} + \mathcal{G}) - \eta \nabla \times \nabla \times \boldsymbol{B}'
\]

\[
\mathcal{G} = \boldsymbol{u}' \times \boldsymbol{B}' - \overline{\boldsymbol{u}' \times \boldsymbol{B}'} \quad \text{usually neglected}, \quad \text{FOSA} = \text{SOCA}
\]

\[
\text{\textbf{B}' linear, homogeneous functional of } \overline{\textbf{B}}
\]

approximation of scale separation: \( \textbf{B}' \) depends on \( \overline{\textbf{B}} \) only in small surrounding

Taylor expansion: \( \left( \overline{\boldsymbol{u}' \times \boldsymbol{B}'} \right)_i = \alpha_{ij} \overline{\boldsymbol{B}}_j + \beta_{ijk} \overline{\boldsymbol{B}}_k / \partial x_j + \ldots \)
Mean-field theory

\[
(u' \times B')_j = \alpha_{ij} B_j + \beta_{ijk} \partial B_k / \partial x_j + \ldots
\]

\(\alpha_{ij}\) and \(\beta_{ijk}\) depend on \(u'\)

homogeneous, isotropic \(u'\): \(\alpha_{ij} = \alpha \delta_{ij}\), \(\beta_{ijk} = -\beta \varepsilon_{ijk}\) then

\[
u' \times B' = \alpha B - \beta \nabla \times B
\]

Ohm’s law: \(j = \sigma (E + (u \times B)/c)\)

\[
\bar{j} = \sigma (\bar{E} + (\bar{u} \times \bar{B})/c + (\alpha \bar{B} - \beta \nabla \times \bar{B})/c) \quad \text{and} \quad c \nabla \times \bar{B} = 4\pi \bar{j}
\]

\[
\bar{j} = \sigma_{\text{eff}} (\bar{E} + (\bar{u} \times \bar{B})/c + \alpha \bar{B}/c)
\]

\[
\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{u} \times \bar{B} + \alpha \bar{B}) - \eta_{\text{eff}} \nabla \times \nabla \times \bar{B} \quad \text{with} \quad \eta_{\text{eff}} = \eta + \beta
\]

Two effects:

1. \(\alpha\) - effect: \(\bar{j} = \sigma_{\text{eff}} \alpha \bar{B}/c\)

2. turbulent diffusivity: \(\beta \gg \eta\), \(\eta_{\text{eff}} = \beta = \eta_T\)
Sketch of dependence of $\alpha$ and $\beta$ on $u'$

\[
\frac{\partial B'}{\partial t} = \nabla \times (\overline{u} \times B' + u' \times B + G) - \eta \nabla \times \nabla \times B' 
\]

simplifying assumptions: $G = 0$, $u'$ incompressible, isotropic, $\overline{u} = 0$, $\eta = 0$

\[
B'_k = \int_{t_0}^{t} \frac{\varepsilon_{klm} \varepsilon_{mrs}}{\delta_{kr} \delta_{ls} - \delta_{ks} \delta_{lr}} \left( \frac{\partial}{\partial x_l} (u'_r \overline{B}_s) \right) d\tau + B'_k(t_0)
\]

\[
\mathcal{E}_i = \langle u' \times B' \rangle_i = \varepsilon_{ijk} \left( u'_j(t) \int_{t_0}^{t} \left( \frac{\partial u'_k}{\partial x_l} \overline{B}_l + u'_k \frac{\partial \overline{B}_l}{\partial x_l} - \frac{\partial u'_l}{\partial x_l} \overline{B}_k - u'_l \frac{\partial \overline{B}_k}{\partial x_l} \right) d\tau + B'_k(t_0) \right)
\]

\[
= \varepsilon_{ijk} \int_{t_0}^{t} \left[ \langle u'_j(t) \frac{\partial u'_k(\tau)}{\partial x_l} \rangle \overline{B}_l - \langle u'_j(t) u'_l(\tau) \rangle \frac{\partial \overline{B}_k}{\partial x_l} \right] d\tau 
\]

isotropic turbulence: $\alpha = -\frac{1}{3} u' \cdot \nabla \times u' \tau^* = -\frac{1}{3} H \tau^*$ and $\beta = \frac{1}{3} u'^2 \tau^*$

$H$ helicity, $\tau^*$ correlation time
Mean-field coefficients derived from a MHD geodynamo simulation

(Schrinner et al. 2007)
Mean-field dynamos

Dynamo equation:
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} + \alpha \mathbf{B} - \eta_T \nabla \times \mathbf{B}) \]

- spherical coordinates, axisymmetry
- \( \mathbf{u} = (0, 0, \Omega(r, \vartheta))r \sin \vartheta \)
- \( \mathbf{B} = (0, 0, B(r, \vartheta, t)) + \nabla \times (0, 0, A(r, \vartheta, t)) \)

\[ \frac{\partial B}{\partial t} = r \sin \vartheta (\nabla \times A) \cdot \nabla \Omega - \alpha \nabla_1^2 A + \eta_T \nabla_1^2 B \]
\[ \frac{\partial A}{\partial t} = \alpha B + \eta_T \nabla_1^2 A \quad \text{with} \quad \nabla_1^2 = \nabla^2 - (r \sin \vartheta)^{-2} \]

rigid rotation has no effect
no dynamo if \( \alpha = 0 \)

\[ \frac{\alpha-\text{term}}{\nabla \Omega-\text{term}} \approx \frac{\alpha_0}{|\nabla \Omega| L^2} \quad \left\{ \begin{array}{c} \gg 1 \quad \alpha^2-\text{dynamo with dynamo number } R_{\alpha}^2 \\ \ll 1 \quad \alpha \Omega-\text{dynamo with dynamo number } R_{\alpha} R_{\Omega} \end{array} \right. \]
Sketch of an $\alpha\Omega$ dynamo

- Poloidal field
  - $\partial\Omega/\partial r < 0$
  - $\alpha \sim \cos \theta$

- Toroidal field by differential rotation;
  - Electric currents by $\alpha$-effect
  - Periodically alternating field, here antisymmetric with respect to equator

- Poloidal field by $\alpha$-effect

- Toroidal field by differential rotation;
  - Electric currents by $\alpha$-effect
Sketch of an $\alpha^2$ dynamo

stationary field, here antisymmetric with respect to equator
MHD equations of rotating fluids in non-dimensional form

### Navier-Stokes equation including Coriolis and Lorentz forces

\[
E \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla^2 \mathbf{u} \right) + 2 \hat{\mathbf{z}} \times \mathbf{u} + \nabla \Pi = Ra \frac{r}{r_0} T + \frac{1}{Pm} (\nabla \times \mathbf{B}) \times \mathbf{B}
\]

- Inertia
- Viscosity
- Coriolis
- Buoyancy
- Lorentz

### Induction equation

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \frac{1}{Pm} \nabla \times \nabla \times \mathbf{B}
\]

- Induction
- Diffusion

### Energy equation

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T + Q
\]

### Incompressibility and divergence-free magnetic field

\[
\nabla \cdot \mathbf{u} = 0 \quad , \quad \nabla \cdot \mathbf{B} = 0
\]
### Non-dimensional parameters

#### Control parameters (Input)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Force balance</th>
<th>Model value</th>
<th>Earth value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rayleigh number</td>
<td>$Ra = \alpha g_0 \Delta T d / \nu \Omega$</td>
<td>buoyancy/diffusivity</td>
<td>$1 - 50 Ra_{\text{crit}}$</td>
<td>$Ra_{\text{crit}} \gg Ra_{\text{crit}}$</td>
</tr>
<tr>
<td>Ekman number</td>
<td>$E = \nu / \Omega d^2$</td>
<td>viscosity/Coriolis</td>
<td>$10^{-6} - 10^{-4}$</td>
<td>$10^{-14}$</td>
</tr>
<tr>
<td>Prandtl number</td>
<td>$Pr = \nu / \kappa$</td>
<td>viscosity/thermal diff.</td>
<td>$2 \cdot 10^{-2} - 10^3$</td>
<td>$0.1 - 1$</td>
</tr>
<tr>
<td>Magnetic Prandtl</td>
<td>$Pm = \nu / \eta$</td>
<td>viscosity/magn. diff.</td>
<td>$10^{-1} - 10^3$</td>
<td>$10^{-6} - 10^{-5}$</td>
</tr>
</tbody>
</table>

#### Diagnostic parameters (Output)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Force balance</th>
<th>Model value</th>
<th>Earth value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elsasser number</td>
<td>$\Lambda = B^2 / \mu \rho \eta \Omega$</td>
<td>Lorentz/Coriolis</td>
<td>$0.1 - 100$</td>
<td>$0.1 - 10$</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>$Re = ud / \nu$</td>
<td>inertia/viscosity</td>
<td>$&lt; 500$</td>
<td>$10^8 - 10^9$</td>
</tr>
<tr>
<td>Magnetic Reynolds</td>
<td>$Rm = ud / \eta$</td>
<td>induction/magn. diff.</td>
<td>$50 - 10^3$</td>
<td>$10^2 - 10^3$</td>
</tr>
<tr>
<td>Rossby number</td>
<td>$Ro = u / \Omega d$</td>
<td>inertia/Coriolis</td>
<td>$3 \cdot 10^{-4} - 10^{-2}$</td>
<td>$10^{-7} - 10^{-6}$</td>
</tr>
</tbody>
</table>

Earth core values: $d \approx 2 \cdot 10^5$ m, $u \approx 2 \cdot 10^{-4}$ m s$^{-1}$, $\nu \approx 10^{-6}$ m$^2$s$^{-1}$
Proudman-Taylor theorem

Non-magnetic hydrodynamics in rapidly rotating system

$E \ll 1, \quad Ro \ll 1$: viscosity and inertia small

balance between Coriolis force and pressure gradient

$$-\nabla p = 2\rho \Omega \times u, \quad \nabla \times (\Omega \cdot \nabla) u = 0$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{motion independent along axis of rotation, geostrophic motion}$$

(Proudman 1916, Taylor 1921)

Ekman layer:

At fixed boundary $u = 0$, violation of P.-T. theorem necessary for motion close to boundary allow viscous stresses $\nu \nabla^2 u$ for gradients of $u$ in z-direction

Ekman layer of thickness $\delta_i \sim E^{1/2} L \sim 0.2$ m for Earth core
inside tangent cylinder: $\mathbf{g} \parallel \Omega$:

- Coriolis force opposes convection
- Outside tangent cylinder:
  - P.-T. theorem leads to columnar convection cells
  - $\exp(im\varphi - \omega t)$ dependence at onset of convection,
  - $2m$ columns which drift in $\varphi$-direction

Inclined outer boundary violates Proudman-Taylor theorem

- Columns close to tangent cylinder around inner core

Inclined boundaries, Ekman pumping and inhomogeneous thermal buoyancy lead to secondary circulation along convection columns:

- Poleward in columns with $\omega_z < 0$, equatorward in columns with $\omega_z > 0$

- Negative helicity north of the equator and positive one south
Convection in rotating spherical shell

\[ \omega_z > 0 \quad \text{and} \quad \omega_z < 0 \]

cyclones / anticyclones

Isosurfaces of positive (red) and negative (blue) vorticity \( \omega_z \)
Taylor’s constraint

\[ 2\rho \Omega \times u = -\nabla \rho + \rho \mathbf{g} + (\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi \]  

magnetostrophic regime

\[ \nabla \cdot \mathbf{u} = 0, \quad \rho = \text{const}; \quad \Omega = \omega_0 \mathbf{e}_z \]

Consider \( \varphi \)-component and integrate over cylindrical surface \( C(s) \)

\[ \frac{\partial \rho}{\partial \varphi} = 0 \] after integration over \( \varphi \), \( \mathbf{g} \) in meridional plane

\[ 2\rho \Omega \int_{C(s)} \mathbf{u} \cdot d\mathbf{s} = \frac{1}{4\pi} \int_{C(s)} ((\nabla \times \mathbf{B}) \times \mathbf{B})_\varphi \, ds = 0 \]  

(Taylor 1963)

net torque by Lorentz force on any cylinder \( || \) \( \Omega \) vanishes

\( \mathbf{B} \) not necessarily small, but positive and negative parts of the integrand cancelling each other out

violation by viscosity in Ekman boundary layers \( \sim \) torsional oscillations

around Taylor state
Benchmark dynamo

\[ Ra = 10^5 = 1.8 \, Ra_{\text{crit}}, \quad E = 10^{-3}, \quad Pr = 1, \quad Pm = 5 \]

radial magnetic field at outer radius
radial velocity field at \( r = 0.83r_0 \)
axisymmetric magnetic field
axisymmetric flow

(Christensen et al. 2001)
Conversion of toroidal field into poloidal field

(Olson et al. 1999)
Generation of toroidal field from poloidal field

(Olson et al. 1999)
Field line bundle in the benchmark dynamo

(cf Aubert)
Strongly driven dynamo model

\[ Ra = 1.2 \times 10^8 = 42 \, Ra_{\text{crit}}, \quad E = 3 \times 10^{-5}, \quad Pr = 1, \quad Pm = 2.5 \]

radial magnetic field at outer radius
radial velocity field at \( r = 0.93r_0 \)
axisymmetric magnetic field
axisymmetric flow

(Christensen et al. 2001)
Comparison of the radial magnetic field at the CMB

GUFM model (Jackson et al. 2000)

Full numerical simulation

Spectrally filtered simulation at

\[ E = 3 \cdot 10^{-5}, \ Ra = 42 \ Ra_{\text{crit}}, \ Pm = 1, \ Pr = 1 \]

Reversing dynamo at

\[ E = 3 \cdot 10^{-4}, \ Ra = 26 \ Ra_{\text{crit}}, \ Pm = 3, \ Pr = 1 \]
Introduction
Basic electrodynamics
Kinematic, turbulent dynamos
Magnetohydrodynamical dynamos
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Dynamical Magnetic Field Line Imaging / Movie 2

Figure 5. Snapshots from (a): DMFI movie 1 of model C and (b): movie 2 of model T. Left-hand panels: top view. Right-hand panels: side view. The inner (ICB) and outer (CMB) boundaries of the model are colour-coded with the radial magnetic field (a red patch denotes outwards oriented field). In addition, the outer boundary is made selectively transparent, with a transparency level that is inversely proportional to the local radial magnetic field. Field lines are also colour-coded in order to indicate \(ez\)-parallel (red) and antiparallel (blue) direction. The radial magnetic field as seen from the Earth’s surface is represented in the upper-right inserts, in order to keep track of the current orientation and strength of the large-scale magnetic dipole. Colour maps for (a): ICB field from \(-0.12\) (blue) to \(0.12\) (red), in units of \((\rho\mu)_{1/2}/\Omega_1D\), CMB field from \(-0.03\) to \(0.03\), Earth’s surface field from \(-210^{-4}\) to \(210^{-4}\). For (b): ICB field from \(-0.72\) to \(0.72\), CMB field from \(-0.36\) to \(0.36\), Earth’s surface field from \(-1.8 \times 10^{-3}\) to \(1.8 \times 10^{-3}\).

Vortices into columns elongated along the \(ez\) axis of rotation, due to the Proudman–Taylor constraint. The sparse character of the magnetic energy distribution results from the tendency of field lines to cluster at the edges of flow vortices due to magnetic field expulsion (Weiss 1966; Galloway & Weiss 1981). Since magnetic field lines correlate well with the flow structures in our models, we will subsequently visualize the magnetic field structure alone. The supporting movies of this paper (see Fig. 1 for time window and Figs 5–9 for extracts) present DMFI field lines, together with radial magnetic flux patches at the inner boundary (which we will refer to as ICB) and the selectively transparent outer boundary (CMB). We will first introduce the concept of a magnetic vortex, which is defined as a field line structure resulting from the interaction with a flow vortex. By providing illustrations of magnetic cyclones and anticyclones, DMFI provides a dynamic, field-line based visual confirmation of previously published dynamo mechanisms (Kageyama & Sato 1997; Olson et al. 1999; Sakuraba & Kono 1999; Ishihara & Kida 2002), and allows the extension of such descriptions to time-dependent, spatially complex dynamo regimes.

3.1.1 Magnetic cyclones
A strong axial flow cyclone (red isosurface in Fig. 4) winds and stretches field lines to form a magnetic cyclone. Fig. 6 relates DMFI visualizations of magnetic cyclones, as displayed in Figs 4 and 5, with a schematic view inspired by Olson et al. (1999). A magnetic cyclone can be identified by the anticlockwise motion of field lines clustered close to the equator, moving jointly with fairly stable high-latitude CMB flux patches concentrated above and below the centre of the field line cluster. Model C (movie 1, Fig. 5a) exhibits very large-scale magnetic cyclones (times 4.3617, 4.3811), which suggest an axial vorticity distribution biased towards flow cyclones. Inside these vortices, the uneven distribution of buoyancy along \(ez\) creates a thermal wind secondary circulation (Olson et al. 1999), which is represented in red on Fig. 6. This secondary circulation concentrates CMB flux at high latitudes, giving rise to relatively long-lived (several vortex turnovers) flux patches similar to those found in geomagnetic field models. Simultaneously, close to the equatorial plane, the secondary circulation concentrates field lines into bundles and also pushes them towards the outer boundary, where CMB flux patches are located. (Aubert et al. 2008)
Reversals

500 years before midpoint  
midpoint  
500 years after midpoint

(Glatzmaier and Roberts 1995)
Reversals

(Aubert et al. 2008)
Power requirement of the geodynamo

- Dynamo converts thermal and gravitational energy into magnetic energy
- Power requirement for geodynamo set by its ohmic losses
- Difficult to estimate: $0.1 - 5 \text{ TW} \sim 0.3 - 10\%$ Earth’s surface heat flow
- Recent estimate from numerical and laboratory dynamos by extrapolating the magnetic dissipation time for realistic values of the control parameters: $0.2 - 0.5 \text{ TW}$ (Christensen and Tilgner 2004)
- Important for thermal budget and evolution of the inner core
- Thermal convection thermodynamically inefficient, compositional convection associated with core cooling
- High heat flow leads to rapid growth of the inner core and low age $\leq 1 \text{ Gyr}$
- Low power requirement estimated by Christensen and Tilgner (2004) is consistent with an inner core as old as $3.5 \text{ Gyr}$
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