Geophysical gravity

1 Earth's orbit in the Solar System

The concept of *Gravity* was introduced to science by Sir Isaac Newton in his *Philosophiae Naturalis Principia Mathematica* in 1687. He recognized that he could explain the periodic orbits of the then-known planets about the Sun by intoducing a very simple concept: the planets are attracted toward the Sun by a gravitational force that is inversely proportional to the square of their momentary distance from the Sun. Newton's Law of Gravitation can be stated in simple mathematical form:

$$\vec{F}_{gravity} = G\,\frac{m_1 \cdot m_2}{|\vec{r}|^3} \cdot \vec{r}$$

where $\vec{r}$ is the vector displacement of one mass, taken to be at the coordinate origin, to the other. Newton recognized that it was *gravity* that provided the *force* required to accelerate each planet along its elliptical orbit.

Let us assign $m_1 = M_{\odot}$ to the coordinate origin centred on our Sun. The force acting on an orbiting planet, mass $m_2 = m_p$, is simply

$$m_p \frac{d^2\vec{r}}{dt^2}$$

which balances the gravitational attraction of the Sun so that

$$\frac{d^2\vec{r}}{dt^2} = -G\frac{M_{\odot}}{|\vec{r}|^3}\vec{r}.$$

While it might not be immediately clear to you, this is a non-linear differential equation that describes the evolution of the orbital position of the planet in time. Non-linear differential equations are not easy to solve by straight-forward analytic manipulations but given initial conditions, we can quite easily model the incrementing position using computers. Digital computers, however, cannot sufficiently resolve the position (due to short numerical word length and round-off error) to obtain an exact solution. Errors accumulate in the recursions.

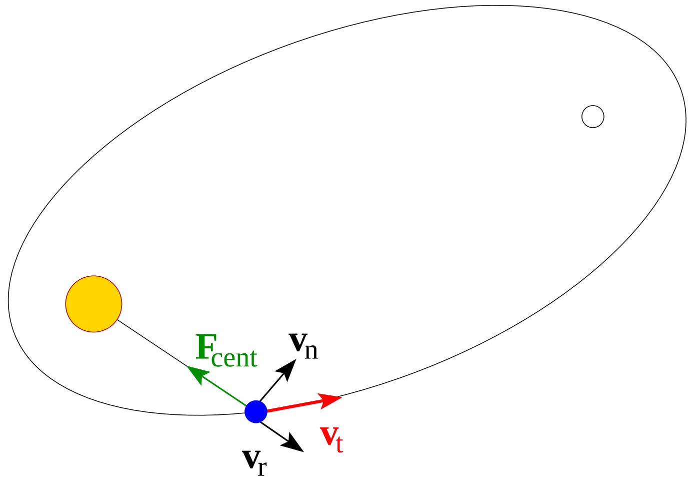

Figure 1 The Sun, sitting at one *focus* of the ellipse of the orbit provides the gravitational attraction, the centripetal force, to hold the planet into its orbit. Without this gravity-provided centripetal force (green vector), the planet would escape, moving with constant velocity, along its orbital tangent (red vector). Note that the other focus is empty.

The velocity of the planet normal to the vector direction of the centripetal force is indicated in black. You might recognize that if the orbit were circular (both foci assembled at the circle's centre), the normal (black) and tangental (red) velocities would be identical.

While the classical mechanics of stable elliptical orbits[1] is quite complicated, if the orbit is circular, it is not. For a circular orbit, $\boldsymbol{v_r} = \mathbf{0}$, $\boldsymbol{v_t} = \boldsymbol{v_n}$ and the planet is always in the same gravitational potential field of the Sun. For elliptical orbits, we have to take into account and balance the accelerations due to the planet's varying distance from the Sun as well as just providing the necessary centripetal acceleration. Let us look at a circular orbit where $\boldsymbol{M_\odot}$ is the mass of the Sun and $\boldsymbol{m_p}$ is the mass of a planet moving in a circular orbit at radial distance $|\vec{\boldsymbol{r}}|$ from the Sun (taken as coordinate origin) with tangential velocity $\boldsymbol{v_t}$. If this orbit is to be stable, the centripetal force,

$$\vec{\boldsymbol{F}}_{\boldsymbol{cent}} = -\frac{\boldsymbol{m_p}|\vec{\boldsymbol{v}}_{\boldsymbol{t}}|^2\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|^2}$$

required to maintain the circular orbit is just that provided by gravity

$$\vec{\boldsymbol{F}}_{\boldsymbol{gravity}} = -\boldsymbol{G}\frac{\boldsymbol{M_\odot m_p}}{|\vec{\boldsymbol{r}}|^3}\vec{\boldsymbol{r}}.$$

[1] Prof. Walter Lewin's lecture on elliptical orbits in his MIT Classical Mechanics course

One might note, then, that for this stable circular orbit, the planet's speed (magnitude of its tangential velocity) depends on the central mass of the Sun, $\boldsymbol{M_\odot}$, and its distance from the Sun, $|\vec{\boldsymbol{r}}|$, and the strength of the gravitational force as measured by the Cavendish constant, $\boldsymbol{G = 6.67 \times 10^{-11}}\ m^3 \cdot kg^{-1} \cdot s^{-2}$:

$$|\vec{\boldsymbol{v}}_{\boldsymbol{t}}| = \sqrt{\frac{\boldsymbol{G\, M_\odot}}{|\vec{\boldsymbol{r}}|}}.$$

If one knows the radius of a planet's orbit and its period (so as to determine its tangential speed in orbit), one can determine the central mass that provides the centripetal acceleration required to hold it in orbit. $\boldsymbol{M_\odot = 1.989 \times 10^{30}}\ kg$. Using just this knowledge and by measuring the tangential speeds of stars in orbit about galaxies, Vera Rubin[2] in 1973 obtained the first observational evidence of *"dark matter"*. Current astrophysical theories of cosmology determine that dark matter comprises about 27% of all the "stuff" of the Universe. Ordinary matter and electromagnetic energy equivalents accounts for somewhat less than 5% of the mass and the mass equivalent of mysterious *"dark energy"* for 68%.

1.0.1 More on the Earth's orbit

Presently, the Earth's orbit is nearly circular with an *eccentricity*,

$$\boldsymbol{e = \sqrt{1 - \frac{b^2}{a^2}} = 0.017}$$

where $\boldsymbol{b}$ is the narrow diameter of the ellipse and $\boldsymbol{a}$, the broad diameter. The eccentricity varies over a period of **413 000** years from an almost perfectly circular orbit, $\boldsymbol{e = 0.005}$, to one with a noticable elliptical form, $\boldsymbol{e = 0.058}$. The mean eccentricity is $\boldsymbol{e = 0.028}$. Moreover the long axis of the ellipse precesses, *apsidal precession*, about the Sun with a period of about **112 000** years.

The Earth's orbital plane defines the reference *ecliptic plane* of the Solar System. Relative to the *invariable plane*, that which is normal to the angular momentum vector of the Solar System, the ecliptic plane oscillates with a period of about **100 000** years. It might not surprise you that on long times scales, all of these orbital motions might well affect the Earth's climate. These cyclical effects on climate, along with several others, were assembled by Milutin Milankovic in the early 1900s to form what is now accepted as the *Milankovic theory*[3] of natural climatological variation.

[2] Vera Rubin discovers "*dark matter*"
Most of our Universe is Missing – a BBC Horizon video

[3] Milankovic theory

1.1 Earth's Moon

Earth's Moon orbits the Earth in **1** month. What is the length of that month? There are many measures of the month depending upon the perspective one takes in determining the period of orbit. The best measure of the orbital period of the Moon for purposes of determining the mass of the Earth about which it orbits is measured in reference to the "fixed stars": **1** *tropical month*. Properly, this orbital period is that about the *centre of mass* of the Earth-Moon system and not about Earth's own centre of mass. For elliptical orbits such as that of the Moon about Earth, the appropriate $|\vec{r}|$ in the equation above is the "length of the *semimajor axis* of the orbital ellipse". This is just half the longest *diameter* through the ellipse. A fully classical mechanical derivation of the the description above would take us to this conclusion. That's too involved for here and too much for me. The length of the *tropical month* is **27.321582241** days of **86400** seconds. Until the advent of atomic clocks, this orbital period offered the most precise measure of time. The semimajor axis of the lunar orbit, measured from the Earth-Moon system's centre of mass is **384748** *km*. The centre of mass of the Earth-Moon system is displaced along a line from the Earth's centre of mass toward the Moon at perigee (Moon closest to Earth during its elliptical orbit.) by **4428** *km* and at apogee (Moon farthest from Earth) by **4943** *km*. Both the Moon and Earth are in elliptical orbits about this centre of mass. The Earth's orbit is entirely within the body of the Earth whose radius is about **6371** *km*.

The Moon's orbital eccentricity, presently $\boldsymbol{e} = \mathbf{0.055}$, the inclination of its orbit relative to the ecliptic plane $\boldsymbol{i} = \mathbf{5.15°}$, the tilt of its rotational axis relative to its orbital axis, **6.69°**, and the inclination of Earth's rotational axis, **23.5°**, conspire to give us varying views of the near-side face of the Moon: Lunar libration. On long time scales, all of these factors vary cyclically.

An exercise (not for grading): I have given you, here, enough theory and data to obtain quite accurate measures of the masses of Earth and Moon. Try to do it!

It is interesting to note that the tropical month is getting longer by **0.000000001506** days ($\mathbf{1.301184 \times 10^{-4}}$ seconds) every year. We shall come to this story later in this section.

Knowing the orbital period of the Moon, the displacement of the Earth-Moon system's centre of mass from the Earth's centre, we can determine the mass of the Earth and the Moon. It is by these means that we have determined the masses of all the planets of the Solar System and of those satellites of planets that probes have encountered. Neither Mercury nor Venus has an orbiting natural satellite, so it wasn't until we had probes pass by these planets that we had accurate measures of their masses. Until we could resolve the 3 major satellites of Pluto and their orbital periods, we had seriously overestimated the size and mass of Pluto. Pluto is smaller and much less massive than Earth's Moon and many of the satellites or moons of the larger planets.

1.2 Lunar tides

The $\mathbf{1/r^2}$ nature of gravitational forces has important implication for the momentary shape of bodies like Earth and Moon. From an Earth perspective, the differential gravity across the diameter of the Earth due to the $\mathbf{1/r^2}$ nature of the gravitational attraction of our Moon lifts and depresses tides across the body of the Earth. That hemisphere of the Earth closest to the Moon is lifted toward the Moon and that opposite the Moon is relaxed relative to the centre of mass of the Earth-Moon system. The gravitational force gradient across the diameter of the Earth amounts to about $\mathbf{3.695 \times 10^{-7}}\ m \cdot s^{-2}$ or to about $\mathbf{3.77 \times 10^{-8}}$ of the gravitational attraction of a mass on the surface toward the Earth's centre. **3.77** parts in $\mathbf{10^8}$ may seem like a very small anomaly in acceleration but the Earth as it spins under the Moon is constantly and locally adjusting to the varying attraction of its internal and surface materials toward the Earth's centre of mass. That side of the Earth closest to the Moon faces a gravity reduction of about **1.8** parts in $\mathbf{10^8}$ compared to the Earth's centre of mass while the opposite side faces a gravity reduction as well. That is, the side closest to the Moon is being pulled toward the Moon; the side opposite the Moon is relaxed away from the Moon. These are the *Earth tides* or *body tides of the Earth*. There are a multiple infinity of body tidal periods; in our own gravimetry work, we used a theoretical tidal model that predicted the **3500** periods with largest amplitudes.

If the Earth were perfectly elastic, it would adjust immediately (actually with the speed of sound within its materials) to these variations in gravity. The Earth, though, is somewhat plastic in rheology and so the adjustment lags the gravitational perturbation along the Earth-Moon line by about **12** minutes.

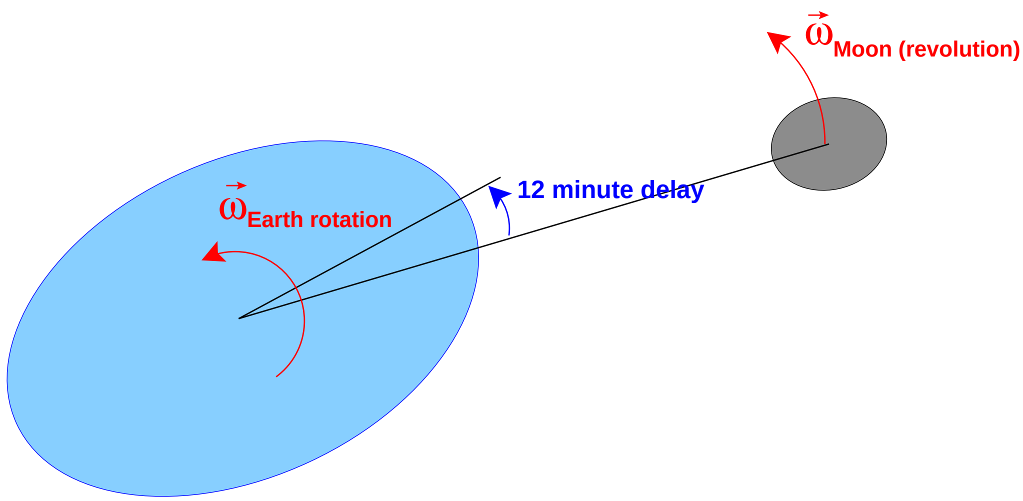

Figure 2 The Moon orbits the Earth in the same sense as Earth rotates on its axis. Earth, however, is rotating very much faster than the Moon's orbital angular velocity. The tidal bulges pulled up on the near-Moon side of the Earth and relaxed from the far-Moon side take some time to be established as the Earth materials deform and some time to relax. This leads to an about 12 minute delay of the tidal bulge from the Earth-Moon line.

As the Earth rotates in same general sense that the Moon orbits about the Earth and with a much higher angular velocity, this means that the *high tide* bulge raised by the Moon has spun forward along the Earth-Moon line. The raised tidal bulge is attracted toward the Moon and so exacts a retarding torque on the spinning Earth. The Earth is slowed in its rotation speed. The *angular momentum* lost from the rotating Earth is taken up by the Moon as a consequence of inviolable *Conservation of Momentum.* The Moon wins angular momentum.

There is another effect, as well, that transfers angular momentum from the spinning Earth to the Moon as a consequence of the tidal forces. On the mobile materials of the Earth, essentially the fluid oceans and atmosphere, the tides raised cause flows of the fluids. The fluids flow but not without a loss of energy in their flowing. Tides affecting the coasts cause mechanical erosion and the movement of grains of sand and rocks back and forth due to the repetitive tidal surgings. Their motions are resisted by friction transferring energy from the mechanical motions of the tides into heat and mechanical damage of the grains and rocks. Mechanical energy is lost in the Earth-Moon system having been degraded into heat and mechanical damage. As it turns out, this mechanical energy loss also contributes to a slowing of the rotation of the Earth. While energy is only conserved in the transferrance of form from mechanical to other forms in this case, momentum is necessarily conserved only in the motions. The Earth slows in its rotation and the Moon gains angular momentum.

The angular momentum, $\vec{L}$, of a point mass, m, moving in a circular path at radius r from the center of revolution is obtained as

$$\vec{L} = m \cdot r^2 \cdot \vec{\omega}$$

where $\vec{\omega}$ is the angular velocity of the motion.

$$|\vec{\omega}| = \frac{2\pi}{T}$$

where T is the period of one rotation.

An exercise (not for grading): Above, I noted by how much the lunar orbital period increases each year. That means that $\vec{\omega}$ is decreasing year by year for the Moon's orbit. The Moon, though is also retreating from Earth as a consequence of the transferrance of angular momentum from the Earth to the Moon. Over the past 40 years, this retreat has been measured to be about $\mathbf{3.85}$ cm/y. Calculate the annual increase in angular momentum of the Moon's orbit. The consequence is that the Earth is losing just this amount of angular momentum annually and so its rotation slows. The *moment of inertia* of Earth referenced to its rotation axis is known to be $I_z = 0.3308 m_{Earth} r^2_{Earth}$[4]. The angular momentum of the Earth is $L_z = I_z \cdot \omega_z$. What is the increase in the length of a day (i.e. 1 rotation of the Earth) in one year that corresponds to the angular momentum transfer to the retreating Moon?

The differential gravity across the body of the Earth due to the Sun's attraction also provides a tide with an average 24-hour cycle. The amplitude of the *solar tide* is about $\mathbf{1/2}$ the amplitude of the *lunar tide*. The tides add together, beating one with the other, to produce a complicated cycle that is only truly periodic over very long time scales.

1.3 Earth rotation: wobbles, nutation, precession

The axis of rotation of the Earth is *nearly* space fixed and aligned with the geographical coordinate system so that $\pm 90^o$ approximately coincides with the rotation axis. Actually, the Earth's rotation is a very complex subject. There are hundreds of effects that disturb this simple model of rotation.

[4] Properly, because the Earth is not exactly rotationally symmetrical, the moment of intertia is only properly and fully expressed as a tensor quantity. Here, I_z is the I_{zz} element of the *moment of inertia tensor*; under rotational symmetry, $I_{xx} = I_{yy}$, the remaining 6 elements of the tensor $\overline{\overline{I}}$ being zero.

The geographical coordinate system was *"inscribed"* on the body of the Earth so as to align in this manner with the rotation axis position averaged during the period January 0, 1900 to December 32, 1905. The average length of day during that period was assigned to be **86400** seconds, so defining the duration of the second. As the Earth's rotation is slowing down (See previous subsection.), the length of day is increasing. Adjustments are made by adding a *leap second* to the year every 1 to 5 years as needed in order to bring the *noon* of atomic-based time in coincidence with the *solar noon*.

The **IERS**, *International Earth Rotation (and Reference Systems) Service*[5] obtains accurate measurements of the Earth's rotation period and publishes them on various intervals, typically, averaged over a period of 5 days. In Figure 3, you see a record of their observations and adjustments to the atomic clock time in coordinating it with the Earth's rotation time so that solar noon is never more than 1 second in variance from the **UTC**, *Universal Coordinated Time*, the standard time that we use both civilly and geophysically.

In 1972, it was decided that rather than continually adjusting our clocks, the leap seconds would be added when necessary during the moment between June 30 and July 1 or December 31 and January 1 each year.

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[5] IERS

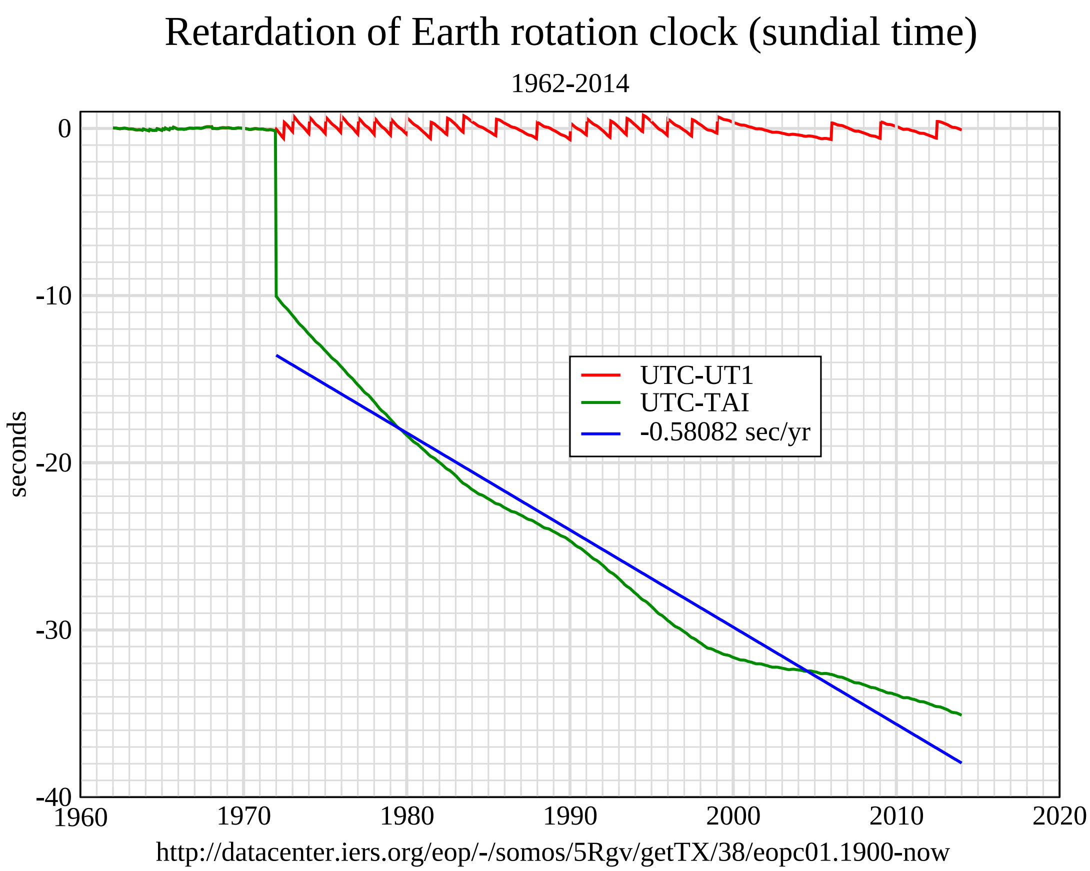


Figure 3: The diagram shows the variance between the time scale coordinated with Earth's rotation (UTC) and that determined by a large suite of atomic time standards distributed in laboratories aroung the world (TAI). This record begins in 1962 and continues through December 31, 2014. Note UTC is "mean solar time" which has taken into account the Earth's orbital eccentricity. Actual sundial noon does not correspond to the accumulation of a sequence of days of **86400** seconds; it depends, as well, on Earth's position during its orbital year[6].

Observatoire de Paris – Earth Rotation Service
Excess length of day

The Earth is a dynamic body with continuing redistributions of mass within its interior and across its surface. Redistributions of mass cause changes in the Earth's

[6] The story of the **analemma**

Inertia Tensor. As angular momentum is conserved in the Earth's rotation apart from the rotational-slowing effect described previously any changes in the Inertial Tensor caused by mass redistributions produce changes in the rotation vector, both in magnitude and direction. Moreover, any spinning body that is not perfectly spherically symmetric throughout can be disturbed into a *free wobble*. This wobble is like that that you may have seen as a child when playing with a spinning top or dradle. Tap the spinning top and it begins to wobble. So does the Earth whenever it's Inertia Tensor or its rotation direction is changed. For example, if it were hit by an asteroid, both its Inertia Tensor due to the addition of mass at a point on the surface and its rotation direction due to the assymmetrical push on the surface would change immediately. The Earth's free wobble is called the *Chandler wobble*[7]; it's period is about **438** days.

The Earth is also affected by forced periodic changes in its Inertia Tensor. On an annual cycle, seasonal meteorology moves mass about the planet. Ice and snow accumulates in the high latitude winter and melts away in its summer. Wind directions and strengths change the surface atmospheric forcing on the body of the Earth. These effects cause the Earth to wobble with an *annual wobble* with a **365.25**-day period. This forced *nutation* beats with the Chandler wobble (a damped free nutuation with a period of $\boldsymbol{\sim}$ **438** days) to produce a modulated cyclical motion of the *Pole Path*. Since the 1900-05 definition of the geographical latitude-longitude coordinate system, Earth's geographical north pole has moved almost **12** m from the present rotation axis!

The rotation pole path and and rotation period has been monitored for over a century. The **ILS**, *International Polar Motion Service*, was established in 1899 to measure the effective change of latitude with time as measured by six observatories distributed around the Earth at $39°08' N$ in order to compute the evolving rotation pole position. This service was updated in the 1960s as new and more accurate geodetic technologies became available, especially **VLBI**, *Very Long Baseline Radio Interferometry*. Now the **IERS**, *International Earth Rotation Service*, coordinates measurements made by all available techniques to produce a pole position averaged over a 5-day period every 5 days.

[7] An excellent article on the subject by D.E. Smylie
Current orientation parameters IERS

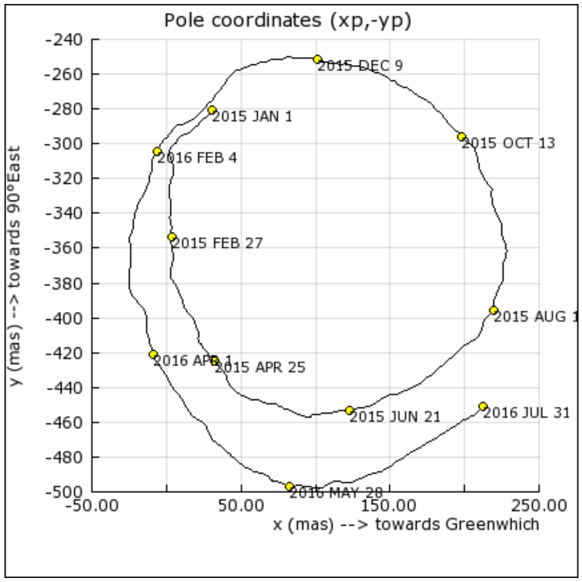


Figure 4: The diagram shows the rotation pole path measured in geographical coordinates starting in January, 2015 and through August, 2016. **1** *mas* (milli-arc-second) represents about **3.09** *cm* Note that, on July 31, 2016, the rotation pole position was about **15.5** *m* south of the CIO (Conventional International Origin) along a longitude line **64°** W. The approximate center of this pole path is oriented **78°** W. Pole-path data from EOC-Paris Observatory
Current-recent position

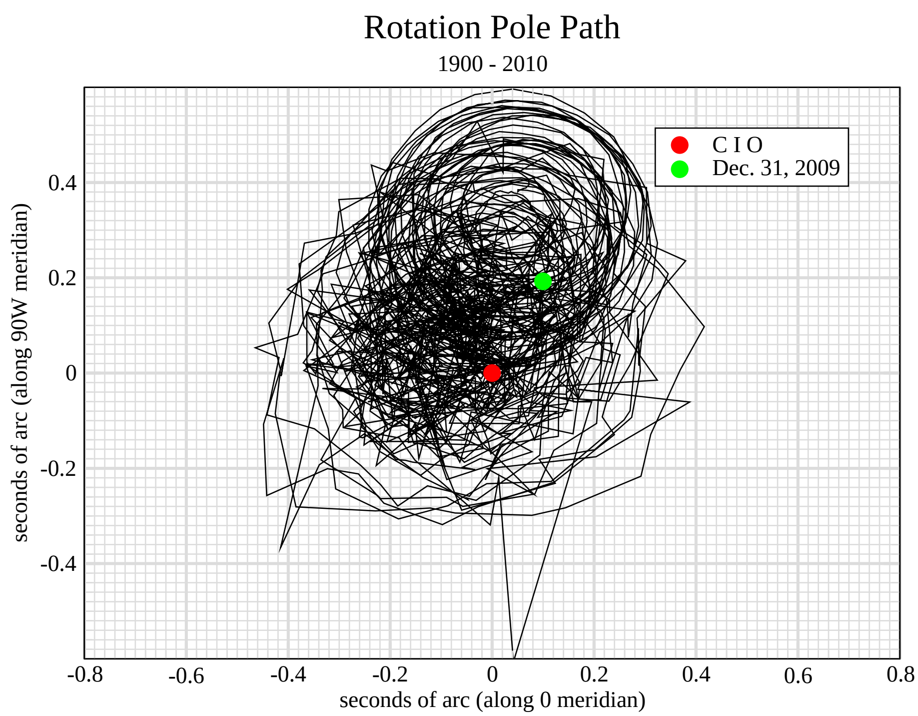


Figure 5: The diagram shows the rotation pole path measured in geographical coordinates starting in January, 1900 and through to December 31, 2009 with temporal increment of $\mathbf{0.05}\,yr$. The CIO (Conventional International Origin) is shown as the red centred dot and the pole position as of December 31, 2009 is shown by the green dot. You might note that the recent centres of the "circles" of motion have drifted about $\mathbf{10}\,m$ south of the CIO along a longitude line about $\mathbf{80°}\,W$ during these 110 years.

Data for plot from EOC-Paris Observatory

In Figure 5, we note that the measurement errors (as evidenced by the recently smooth track) have been incredibly reduced by the new technologies. Presently, standard errors in VLBI measurements, those most heavily weighted in the pole-position 5-day measurement, have not exceeded about $\mathbf{3}\,mm$ since 1990, actually less than the width of the line in the diagram. In the early 1900s, errors of several metres colour the data.

From these measurements, we can obtain the periods of the cyclical motion due to the **Chandler Wobble** (i.e., the Earth's free Eulerian wobble) and another which is driven by annual meteorological forcing. The spectrum of polar motion cycles is rich in many other geophysical effects.

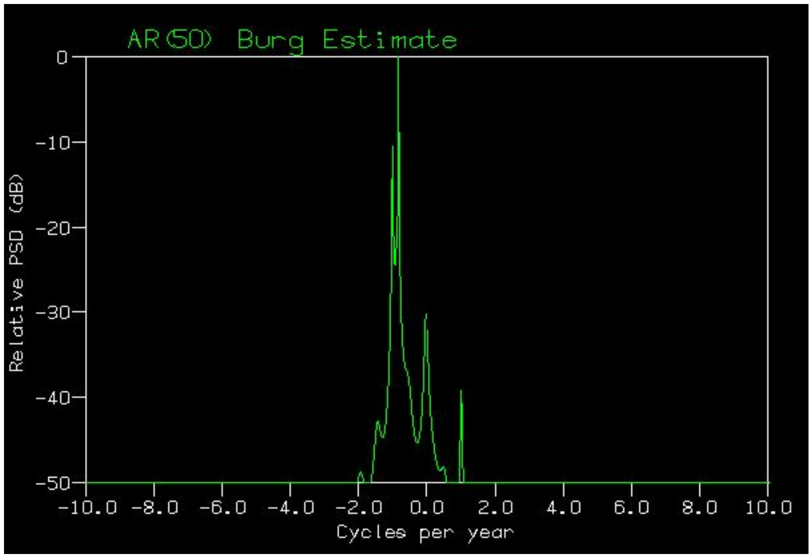


Figure 6: The diagram shows a high-resolution "maximum-entropy" power spectrum of the rotation path for the years 1990-2009. Note the rotation direction of the path is largely described by a negative frequency (representing clockwise motion in the coordinate system of the measurements), here measured in cycles/per year. The highest peak is that of the **Chandler Wobble** whose period is found to be about **438** *days*; that peak just left of the Chandler represents the residual (following our corrections – see below) forced annual wobble.

Data used in this plot from IERS

In order to produce this spectrum from the pole-path data set, many corrections based on good theory and past experience were necessarily applied. A low-frequency background drift was removed, then a "best-fitted" model of the annual component of the pole path which, in this case, left us with a residual time series that retained the Chandler free wobble and some other lower amplitude wobbles and nutations. It is this Chandler wobble that might be excited by earthquakes, asteroid impacts and other geophysical effects. The annual component does dominate the pole path record but is, in principle, less than a factor of 2 greater in amplitude. We remove it as best we can in order to see through to the Chandler component.

1.4 Precession

As well as the body of the Earth moving across the rotation axis of the Earth, the rotation axis itself moves relative to the inertial system of the *"fixed stars"*. Properly, the inertial reference system is determined by the positions of the most distant quasars on the *Celestial Sphere*. These are tied to the various geographical reference systems for the Earth through *"very long baseline radio interferometry" (VLBI)* involving many of the largest radiotelescopes on Earth. Even before this extremely high accuracy measurement technique, the path of the rotation axis of the Earth was recognized and monitored as the apparent movement of the *"pole star"*, Polaris. Presently, Polaris that bright star most closely aligned with Earth's rotation axis; in about **13000** years, the rotation axis will have moved to become almost coincident with the bright star Vega.

The *Precession* of the rotation axis is driven by torques applied to the rotating Earth due to differential gravitational forces imposed by the Moon on Earth's equatorial bulge.

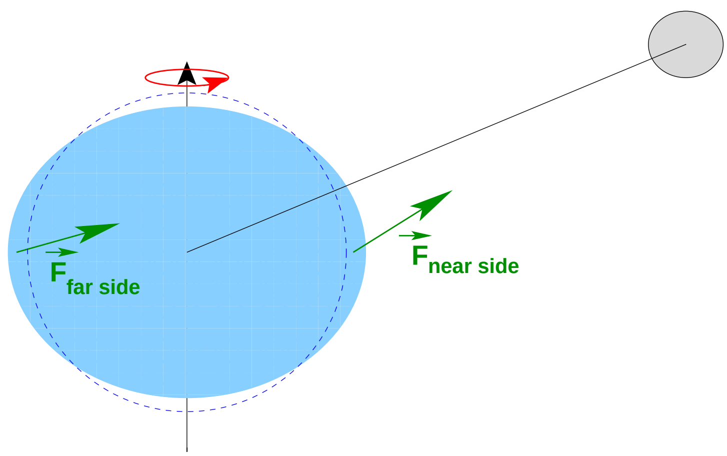


Figure 7: The difference of the gravitational attractive forces toward the Moon acting on the equatorial bulge on the near-Moon side of the Earth and far-Moon side produce a torque on the rotating Earth. The Earth's reaction to the torque produces a continuous, **25772**-year precession of Earth's rotation axis.

1.5 Shape of the Earth and surface gravity

If the Earth were isolated, homogeneous, at least layer by layer, non-rotating and in equilibrium, it would be pulled into a perfectly spherical shape through gravity. It is in fact almost spherical but it is rotating once every **86164.092** seconds relative to the inertial system of the fixed stars. This is one *"siderial day"*. One *"solar day"* corresponds to **86400** seconds because, each day, the Earth has to spin just **1/365.25** of one full rotation just to return to the same perspective toward the Sun – that is, from one noon to the next. The Earth rotates **366.25** times during one year of **365.25** solar days. The stars are so far away that our perspective on the stars doesn't much change during a whole year. Actually, for the nearby stars, it does change just a little and we can use this fact to determine the distance to nearby stars but that's not the topic of this course nor of this subsection of this note.

If the Earth were non-rotating, perfectly spherical and with polar radius $\boldsymbol{r = r_p}$, the gravitational acceleration everywhere on the surface would be determined by Newton's law of gravitation:

$$\vec{g_N} = -\frac{G\,M_{\male}}{|\vec{r}|^3}\vec{r}.$$

This is essentially the gravitational attraction at the Earth's poles at *"sea level"*. However, as the Earth is rotating, and as we move from the poles toward the equator, we require evermore centripetal acceleration to hold us toward the Earth's centre. The effect is that the acceleration that we feel toward the centre decreases as we move from poles to equator. In this simple model, the Newtonian gravity provides the centripetal acceleration as

$$\vec{a_{cent}} = -|\vec{\omega}|^2\vec{r_r}$$

where the vector $\vec{r_r}$ is the vector distance from the rotation axis. At any latitude on the rotating Earth, the total acceleration felt by a mass on the surface is then $\vec{g}_N = \vec{g}_T + \vec{a}_{cent}$. While $\vec{g}_N$ is many times larger than $\vec{a_{cent}}$, at the equator, their directions are aligned with each other so that at the equator, the acceleration toward Earth's centre is just the difference in magnitudes. At the poles, $\vec{a_{cent}}$ is zero and so only the Newtonian gravitational acceleration holds. However at all other latitudes, the vector directions of $\vec{a_{cent}}$ and $\vec{g}_N$ must be added. Figure 7 shows that upon this addition of the two acceleration effects, at latitudes other than O^o and $\pm 90^o$, the vector direction of "down" does not point precisely to the Earth's centre but rather to a point on the rotation axis in the opposite hemisphere.

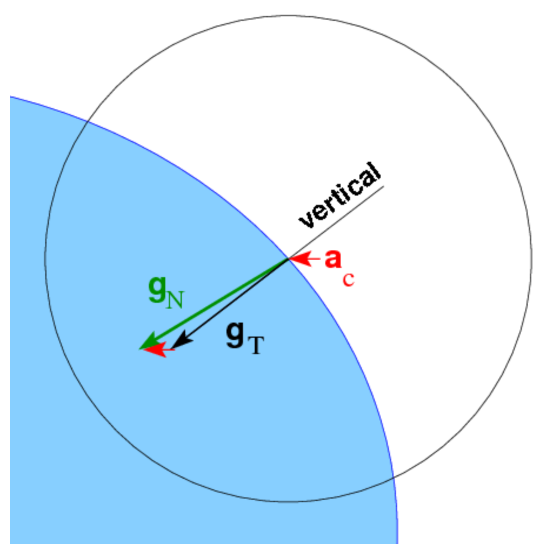


Figure 7: The effect of the centripetal acceleration on the total downward acceleration and the direction of the vertical.

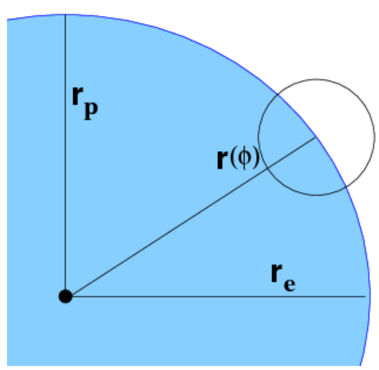
What has been left from this analysis is the fact that the Earth's body and surface adjusts to the lowered, downward vertical acceleration at the equator as well and so the radius of Earth's equator is greater than the polar radius by about $\mathbf{1}$ part in $\mathbf{298.25}$, the *"flattening parameter"*:

$$f = \frac{r_e - r_p}{r_e}.$$

In order to take into account the full story of the variation in gravitation acceleration over the surface of the Earth and the flattened ellipsoidal shape of the Earth is, I think, beyond the scope of this U2 course. A proper, but still approximate analysis was first accomplished by Alexis Claude Clairault[8] in 1743. The limited argument I have given is essentially that of Newton, a century earlier. Clairault's analysis is known as Clairault's Theorem[9] which is regarded as one of the major accomplishments in geophysical theory. Clairault's analysis obtains the *radius* of the Earth as a function of latitude as $r(\phi) = r_e(1 - f\sin^2(\phi))$ where r_e is the equatorial radius and the *flattening*

$$f = \frac{3(I_{zz} - I_{xx})}{2r_e^2 M_{\oplus}} + \frac{m}{2}$$

where m is the ratio of the centripetal acceleration at the equator to the Newtonian gravitational attraction at the equator. The IERS'[10] best current estimate for $f = 1/298.25642 \pm 0.00001$ corresponding to $r_e = 6,378,136.6\,m$ and $r_p = 6,356,751.9\,m$.



8 Alexis Claude Clairault
9 Clairault's Theorem
10 Earth Rotation Service's *General Definitions and Numerical Standards*

Figure 8: Flattening and the shape of the Earth.

On the *reference ellipsoid* which most closely corresponds to the *geoid*, $\vec{g_T}$ is defined, geodetically, as the acceleration, locally downward: $\vec{g_T} : [g_{T_x} \;\; g_{T_y} \;\; g_{T_z}] = [0 \;\; 0 \;\; g_{T_z}]$. The *International Gravity Formula (1967) – Helmhertz' equation* obtains g_{T_z} as a function of latitude in accord with Clairault's theorem,

$$g_{T_z} = 9.780327(1.0 + 0.0053024 \sin^2(\phi) - 0.0000058 \sin^2(2\phi)) \;\; [m \cdot s^{-2}],$$

where ϕ is the local latitude.

1.5.1 The geoid

The *geoid* is that equipotential surface that most closely corresponds to the equilibrium, hydrostatic, nearly ellipsoidal shape of Clairault's Earth. Colloquially, it is the *mean sea level* datum surface. Being an equipotential surface, there is no locally horizontal acceleration on that surface. The gradient of equipotential defines the acceleration on that surface, $\nabla U = \vec{g_T}$, where $\vec{g_T}$ assembles the sum of the Newtonian mass effect and the centripetal acceleration effect as well as effects of any internally anomalous mass-density within the Earth.

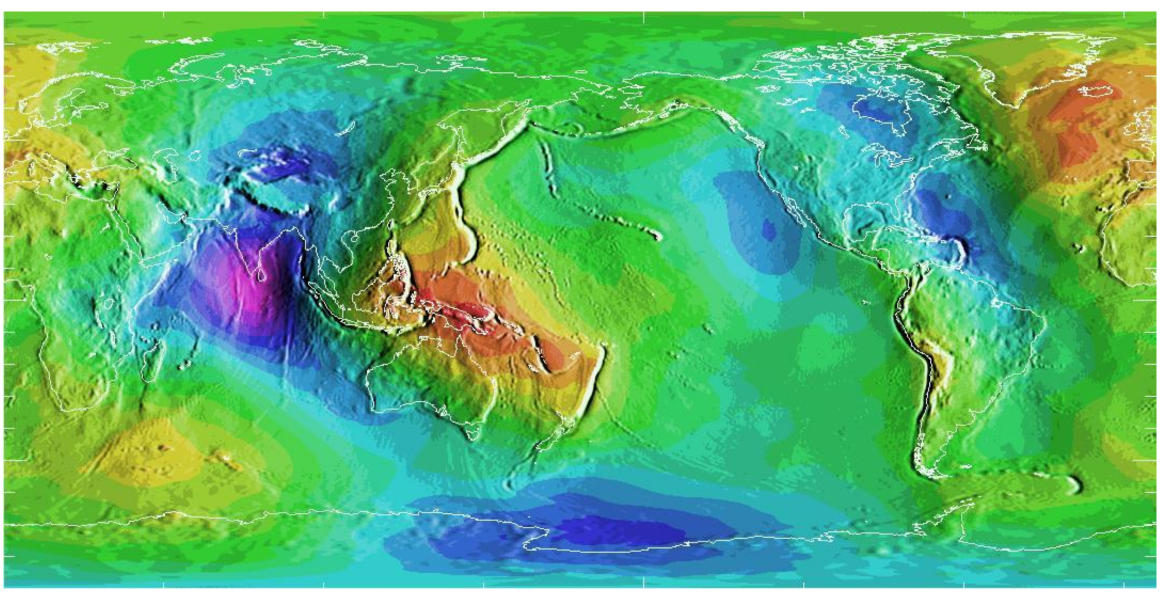


Figure 9: The geoid shown as departures from the reference ellipsoid. The deepest departure, south of India, is approximately **107** metres below the ellipsoid and the highest departure in New Guinea is about **85.4** metres above the reference ellipsoid.

Map from US Naval Academy based on NASA/GSFC data[11].

The undulations of the geoid can tell us quite a lot about the mass distribution within the Earth, about tectonic processes and about convection in the deeper mantle[12]. You should also note that the gravitational **potential** is constant on the geoid; it is not a surface of constant gravitational **acceleration**.

1.5.2 Interpreting gravity I:[13]

- **The geoid and isostasy:** Recognizing that the geoid is an equipotential surface and that the gravitational potential function due to a mass descreases as we approach the mass according to convention, we note that where the geoid surface is low and hence closer to the Earth's centre, it adjusts to that level because mass is deficient in the sectoral column. That is, there is mass deficiency beneath geoidal lows and mass excess below geoidal highs. If the Earth were composed of uniform layers of materials and in hydrostatic equilibrium, the geoid would conform to the reference ellipsoid. That it doesn't allows us to learn something about internal mass distribution.

  The reasons for various low regions of the geoid differ. For example, that over northern Canada is due to depression of the surface by the Laurentian icesheet that melted away as recently as **8000** years ago. The mantle's rheology is plastic-like and the mass, mostly below the lithoshere, that was forced away by the glacial load has not yet flowed back under that region. The crust of the Earth is still rising at almost $\mathbf{2}\, cm/yr$ around James Bay and Hudson's Bay. Coastlines, there, are rising and retreating. This is due to *isostatic adjustment*.

[11] Source site for geoid map.
[12] Geological interpretation of geoidal variations
[13] Interpreting gravity

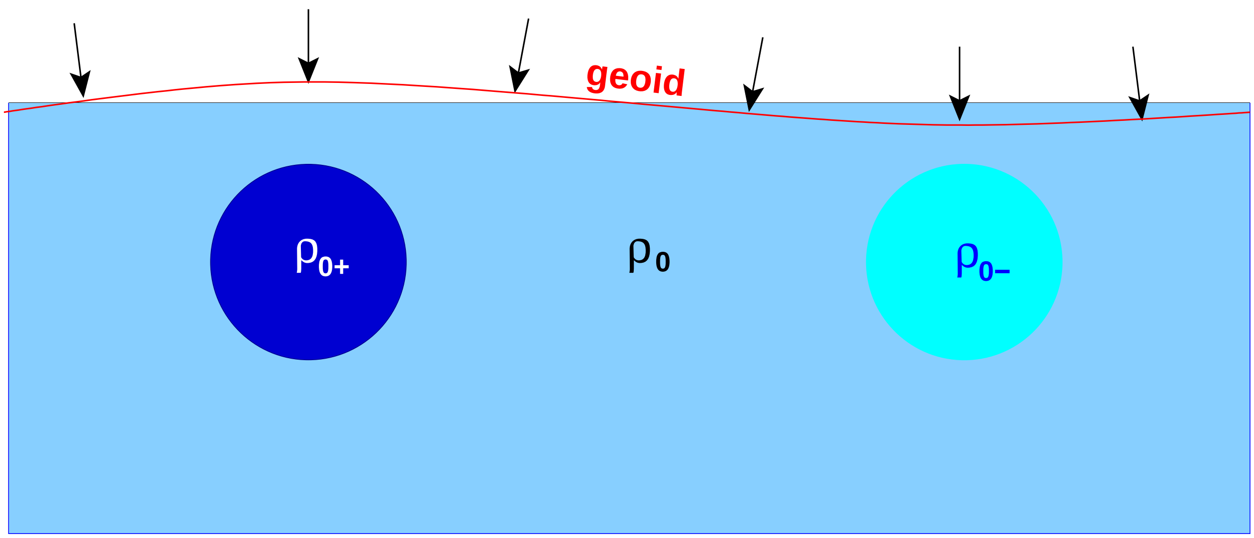
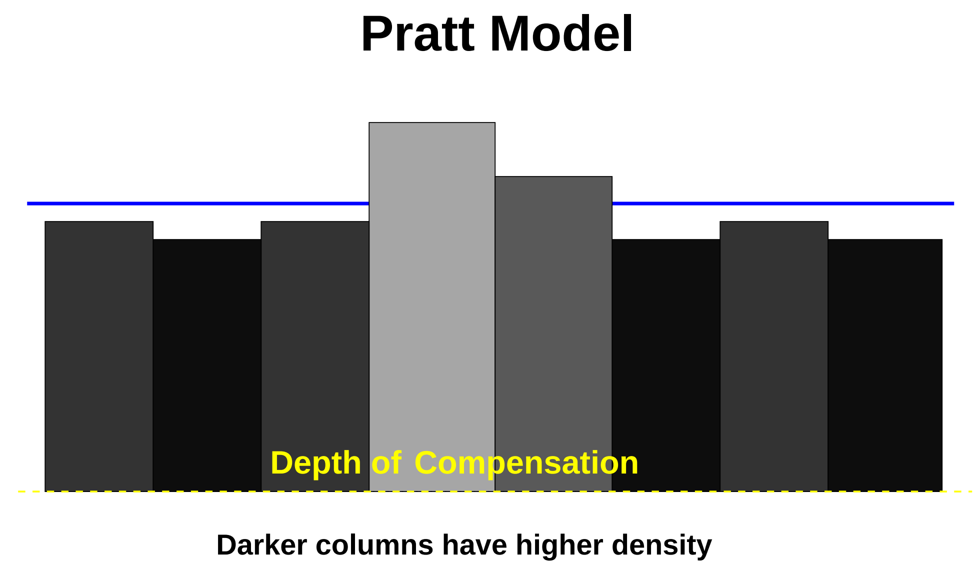
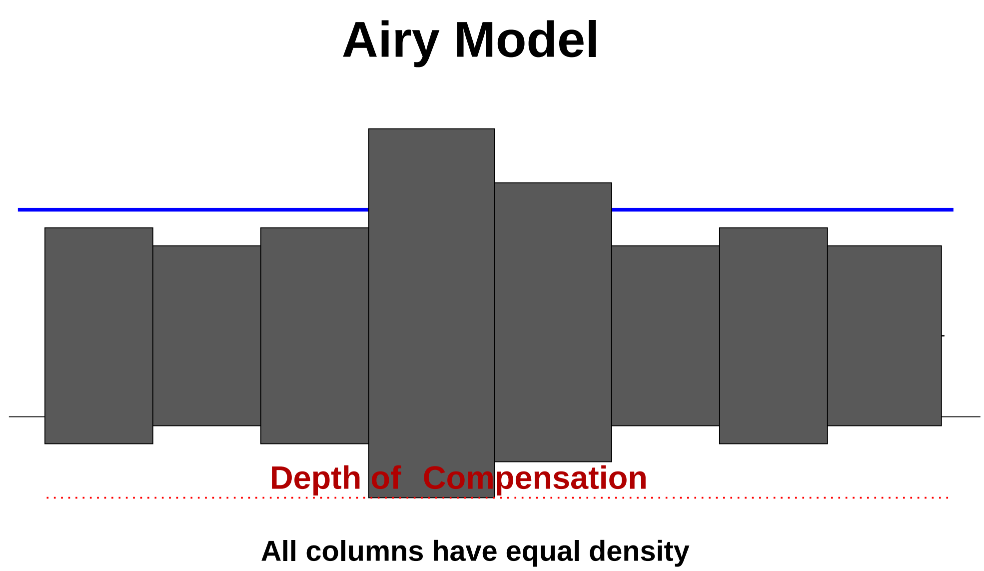


Figure 10: Geoidal surface above high $\boldsymbol{\rho_{0+}}$ and low $\boldsymbol{\rho_{0-}}$ mass densities. Note the gravitational acceleration vector direction is normal to the geoidal surface. Note, as well, that the gravitational acceleration vector is not a constant on the geoidal surface but that it's horizontal components are $\mathbf{0}$.

Before continuing with the interpretation of gravity measurements, it would be useful to understand *isostasy*.

- **Isostasy:** By the mid-1800s, the reasons for topography – differences in elevation – on the Earth were not yet understood. George Everest had by 1830 recognized that there was something odd about the mass of the Himalayan Plateau when he was correcting his survey measurements across India. We shall deal with his problem later even though his argument for its solution might have already have provided a prior explanation to the question posed to the members of the Royal Society of London: *"What is the explanation for the elevation of the Himalayan Plateau and the Ande's Mountain chain?"*. In 1855, two models were offered to the members of the Society in explanation. Archdeacon **J.H. Pratt** suggested that the reason for high elevations is that light materials "float" higher than do dense materials and that the rock of areas of high elevation are of low density. **G.B. Airy** proposed another model: the high-standing regions are compensated by deeper roots but their densities are similar to those of low stands. Airy's hypothesis accounts, for example, for the height of icebergs. An iceberg that floats high has great root depth. We now know that over the continental regions of the Earth both models contribute almost equally to topographic variations. Variations in ocean bottom depth is largely explained by Airy's model. That is, where the oceanic lithosphere is thickest, and hence oldest, it sinks deeper.





1.5.3 Measuring gravitational acceleration

Sensitive gravimeters are easily capable of measuring variations in gravity equivalent to $\mathbf{1}$ part in $\mathbf{10^7}|\vec{g}|$ on the surface of the Earth. That is, we easily and accurately measure to the $\mathbf{7^{th}}$ decimal place in the surface gravitational acceleration. Geodetic instruments can be more sensitive by a factor about **100** and contemporary stationary observatory instruments by a factor of well more than **1000**. At such details of measurement, rather minor variations in elevation, local subsurface density, Earth tides and even tidal and barometric variations in the overlying atmosphere become important. We must correct these effects away in order to recognize and interpret the contributions to gravitational accelerations that interest us.

- **Elevation – free-air correction:** You would not be surprised to learn that as we increase our elevation above the reference ellipsoid determined by Clairault's theorem and Helmhertz' equation, gravity decreases due to the increasing distance from Earth's centre of mass. If we were to measure gravity at elevation $\boldsymbol{h}$ m above the reference ellipsoid, gravity would decrease by $\boldsymbol{\partial_z(g_{T_z})\cdot h}$. We correct to the reference ellipsoid datum by adding in this amount to our observation of gravity at elevation in a *free-air correction.* On the geoid, that equipotential surface most closely fitting the reference ellipsoid, $\boldsymbol{\partial_z(g_{T_z}) = 3.086 \times 10^{-6}}$ $m \cdot s^{-2}/m$.

- **Elevation – Bouguer density correction:** Typically, when making gravity measurements, we are at elevation because we are on a plateau, hill or mountain. We seldom make measurements of gravitational acceleration from the air because we have no stable from which platforms to do so. We recognize that the intervening mass of the hill or mountain from the level of the reference ellipsoid datum has mass and that that mass is gravitationally attractive. An observed measure of gravity is increased by $\boldsymbol{2\pi G\rho_b h}$[14] where $\boldsymbol{\rho_b}$ is the so-called Bouguer density of the intervening crustal rock mass. Typically, using $\boldsymbol{\rho_b = 2700}$ $kg \cdot m^{-3}$, $\boldsymbol{2\pi G\rho_b = 1.132 \times 10^{-6}} m \cdot s^{-2}/m$. We subtract this amount from an observation to take the free-air corrected measurement to the reference ellipsoid datum when the intervening material is not *free air.*

  Often you will see gravity data provided as Free-air gravity or as Bouguer gravity wherein a datum value is that of the theoretical gravity that one would expect on the reference ellipsoid.

- **Terrain correction for local hills and valleys:** Local highlands above the level of measurement produce a gravitational attraction which reduces the downward gravitation. Also, local valleys and lowlands below the level of measurement reduce the downward attraction. Both local hills and valleys reduce the measured gravity. This is taken in account in gravity measurement by modelling their contribution and then adding resultant and compensating downward acceleration component.

[14] Properly, this correction holds for elevation above a plane and not for the whole spherical Earth. If one were to consider a layer of thickness $\boldsymbol{h}$ covering the whole spherical Earth, the correction would be $\boldsymbol{4\pi G\rho_b h}$. Normally, though, we are concerned with only local elevations when making this correction and then the stated value is more nearly the correct one.

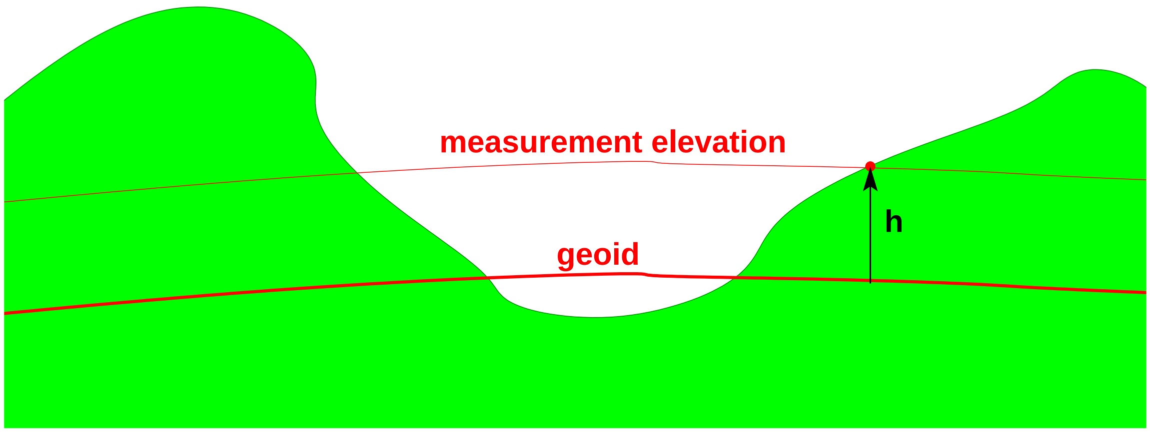


Figure 11: Corrections to gravity measurements made at elevation from the geoid — a work chart!

Upon applying all three of the these *corrections*, we can simulate the acceleration of gravity that would be measured on a flat plane at the elevation of our measurement. We would expect that, if there is no nearby anomalous masses, our corrected gravitational acceleration measurements would be idential for all of our measurements. Particularly in geophysical prospecting for local near-surface masses that might indicate mineralization, reservoirs of petroleum, or variation in subsurface geological structure, these corrections are always applied. For geodetic and global geophysical gravity work, either the geoid which is best measured by satellite orbital perturbation or the free-air and/or Bouguer *anomaly* are most useful. Recently, gravitational accelerations are being measured at satellite elevation[15]. Using the fact that where no intervening sources of mass contribute to the field, Laplace's equation, $\boldsymbol{\nabla^2 U(\vec{r}) = 0}$, where $\boldsymbol{U(\vec{r})}$ is the gravitational potential at the satellite's orbit, we can *downward continue* the gravitational potential and acceleration measurements to the surface or geoid. Note that $\boldsymbol{\nabla(\nabla^2 U(\vec{r})) = \nabla^2 \vec{g}(\vec{r}) = 0}$ also.

[15] The GRACE program
Grace measures gravity variations
ESA's GOCE program

- **Colloquial units in gravitational acceleration measurements:**
  $\mathbf{1}Gal = \mathbf{10^{-2}}\ m \cdot s^{-2}$; $\mathbf{1}\ g.u.\ (gravity\ unit) = \mathbf{10^{-4}}\ Gal$.

1.5.4 Modelling gravity anomalies:

The gravitational potential at position $[\,\mathbf{0\ \ 0\ \ 0}\,]$ due to a mass element of volume $\boldsymbol{dv = dx \cdot dy \cdot dz}$ and density anomaly $\boldsymbol{\rho_c}$ at position $[\,\boldsymbol{x\ \ y\ \ z}\,]$ is obtained as

$$\boldsymbol{dU}(0) = -\frac{\boldsymbol{G\rho_c}}{\sqrt{\boldsymbol{x^2 + y^2 + z^2}}}\boldsymbol{dv}.$$

While the coordinate system might not have been well chosen for the particular problem, we could quite easily build up any extended mass structure using the differential bricks of volume $\boldsymbol{dv}$.

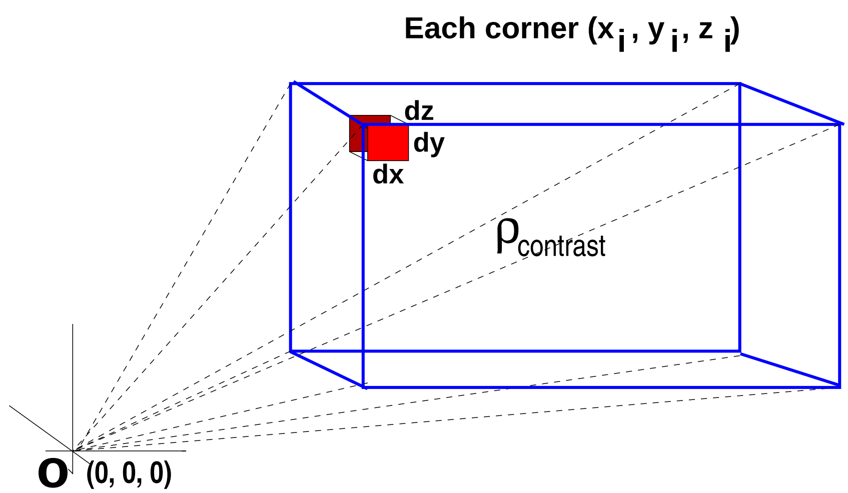


Figure 12: Modelling a gravitational anomaly.

For the extended body shown in Figure 12, we easily(!) obtain:

$$\boldsymbol{U}(0) = \int_{\boldsymbol{V}} \boldsymbol{dU}(0) = -\int_{\boldsymbol{x_1}}^{\boldsymbol{x_2}}\int_{\boldsymbol{y_1}}^{\boldsymbol{y_2}}\int_{\boldsymbol{z_1}}^{\boldsymbol{z_2}} \frac{\boldsymbol{G\rho_c}}{\sqrt{\boldsymbol{x^2 + y^2 + z^2}}}\boldsymbol{dxdydz}.$$

Integrating this over range $\boldsymbol{z_1 \longrightarrow z_2}$ then this result over range $\boldsymbol{y_1 \longrightarrow y_2}$ and finally that result over range $\boldsymbol{x_1 \longrightarrow x_2}$ will be quite laborious even when using the convenience of the WolframlAlpha service. This gives us the gravitational potential due to that block seen from the coordinate origin. To obtain the measurable, gravitational acceleration, we would then have to compute the gradient of the computed potential:

$$\vec{\boldsymbol{g}}(0) = -\boldsymbol{\nabla U}(0).$$

If it were that we wanted the gravitational acceleration rather than the potential, we could obtain this directly by another route.

Just as the gravitational potential contributions due to each elemental brick add up to give us the full potential, so the gravitational acceleration contributions also add up brick-by-brick. Unfortunately, though, $\vec{g}(\mathbf{0})$ is a vector quantity and so we would have to solve for each of the components of the vector separately. This could lead to quite an elaborate sequence of computations and as I claim that I want you to understand principles rather than details through calculation, we won't go there today.

Providing one is not very close to the *block*, useful approximations can be obtained for the gravitational potential and the measurable gravitational acceleration at our origin. One might, for example, replace the full mass of the block by an equivalent mass at its centre of mass:

$$\vec{r}_{com} : [\, \frac{x1 + x2}{2} \;\; \frac{y1 + y2}{2} \;\; \frac{z1 + z2}{2} \,].$$

Being a little fussier, we might divide the box into 8 sub-blocks and obtain a better approximation of the potential and acceleration at our origin. We could divide each side into 3s and now with 27 sub-blocks obtain an even better approximation. These latter calculations are best performed numerically.

For the calculation based on the single mass replacement, though, the issue is simple enough to compute. For algebraic convenience, let us substitute $X = (x1 + x2)/2$, $Y = (y1 + y2)/2$ and $Z = (z1 + z2)/2.$ The radial distance to that center of mass is

$$|r_{\vec{com}}| = \sqrt{X^2 + Y^2 + Z^2};$$

the vector direction is determined simply as the unit length vector – let us call it $\hat{u}_{r_{com}} = \frac{\vec{r}_{com}}{|\vec{r}_{com}|}$. The components of this unit vector are the *direction cosines* of the vector $\hat{u}_{r_{com}} : [\, \alpha \;\; \beta \;\; \gamma \,]$, where:

$$\alpha = \cos a = \frac{X}{|\vec{r}_{com}|},$$

$$\beta = \cos b = \frac{Y}{|\vec{r}_{com}|},$$

$$\gamma = \cos c = \frac{Z}{|\vec{r}_{com}|}$$

and $\alpha^2 + \beta^2 + \gamma^2 = 1.$ See the diagram of Figure 13.

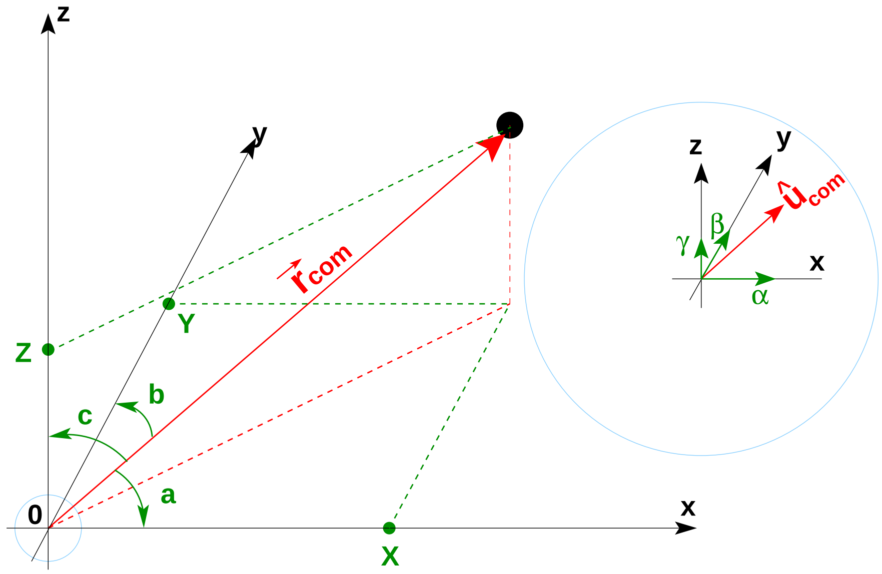


Figure 13: Locating the COM (centre of mass) of the block.

The direction cosines determine the direction of gravitational acceleration anomaly due to the mass: $[\,\boldsymbol{\alpha}\ \ \boldsymbol{\beta}\ \ \boldsymbol{\gamma}\,]$. We calculate the magnitude of gravitational acceleration toward the block and scale its components by the direction cosines of the unit vector.

The volume of the block is $\boldsymbol{V = |x2 - x1| \cdot |y2 - y1| \cdot |z2 - z1|}$; its mass is $\boldsymbol{M_a = \rho_c V}$ and the magnitude of the acceleration toward that block is then,

$$|\vec{g}_a| = \frac{\boldsymbol{G M_a}}{|\vec{r}_{com}|^2}$$

with each of the three components of the vector acceleration scaled by the direction cosines.

$$\vec{g}_a : [\,|\vec{g}_a|\boldsymbol{\alpha}\ \ |\vec{g}_a|\boldsymbol{\beta}\ \ |\vec{g}_a|\boldsymbol{\gamma}\,].$$

- **George Everest's survey transect across India:** Colonel Sir George Everest[16] was Surveyor-General of India from 1830 to 1843. He completed the Great Trigonometric Survey of India along a meridian arc from the far south of the subcontinent north to Dehradun and Nepal. As he surveyed, he very accurately *chained* the distance along the meridian, about **2400** kilometres. He also employed navigation by the stars to measure the meridianal distance. He found a discrepancy between the measurements of the differences of astronomic and geodetic latitudes of Kalianpur on the Ganges plain and Kaliana, near

[16] George Everest
The Great Trigonometric Survey

Dehradun[17]. His geodetic distance corresponded to a lesser latitude difference of **5.24″**. He correctly attributed the discrepancy to a small difference to a deflection of the plumb bob that defined the local vertical as he approached Kaliana, near Dehradun, and the Himalayan Plateau. Pratt attempted to determine the difference based on his model for isostasy; he calculated that the gravitational attraction to the Himalayan Plateau should have deflected the plumb line by **15.885″**, three times as much. Airy recognized that the overestimate was due to mass-deficient roots of the mountains in accord with his model for isostasy.

Pratt used essentially, the gravitational anomaly model above, in his overestimation of the deflection.

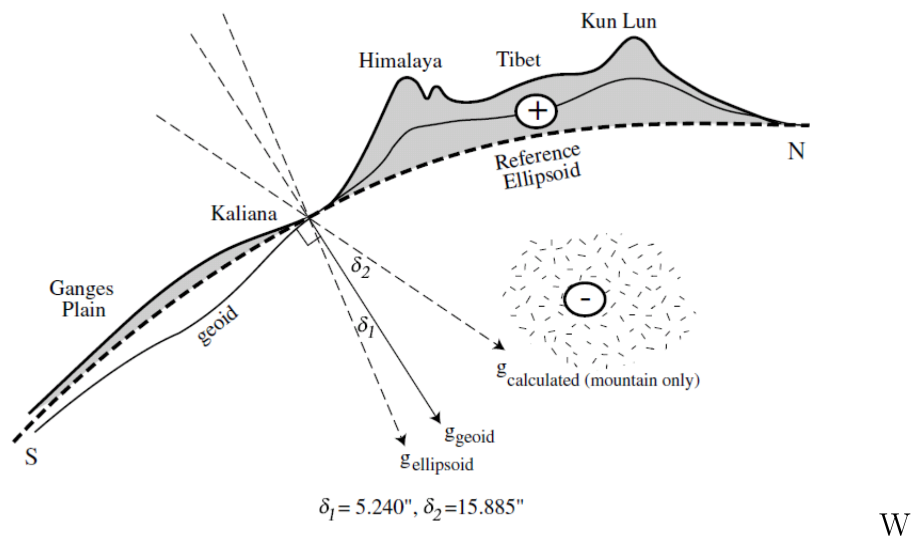


Figure 14: Everest's anomalous deflection of the vertical... After A.B. Watts.

An exercise (not for immediate grading but recommended as prototype for the first take-home assignment): Using the very simple model above with replacement for the point mass at the centre of mass, calculate the horizontal deflection of Everest's plumb bob (level) at Dehradun, India due to: **a)** the mass of the Himalayan Plateau ignoring the mountain roots and **b)** in consideration of the mountain roots.

- **Regional levelling of gravity surveys[18]:** Airy isostasy tends to produce a reduction of gravity (acceleration measures) as seen in broad regional surveys

[17] From Kalianpur to Kaliana: 1100 kilometres
[18] Applied Geophysics: Gravity Theory

over highlands and plateaus. While most surveys employing gravitational acceleration are not focussed on regional surveys and the free-air, Bouguer and terrain corrections reduce those effects that are not interesting to our survey, another correction, that recognized by Everest and Airy, is required to *level* surveys over regional scales.

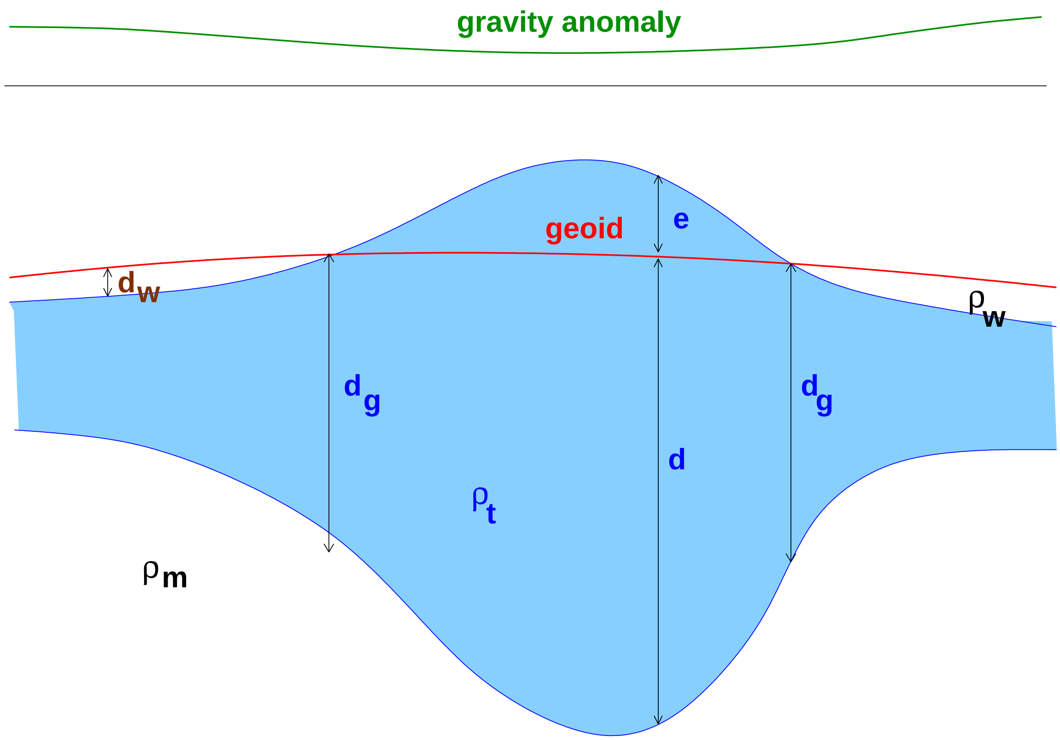


Figure 15: Isostatic correction modelling for regional surveys. $\boldsymbol{d_w}$: depth of ocean basin, $\boldsymbol{e}$: elevation of topography, $\boldsymbol{d_t}$: depth to root (calculated under Airy assumption), $\boldsymbol{d_g}$: depth to root at point where elevation corresponds to geoid (usually taken to be **33** kilometres), $\boldsymbol{\rho_m}$: mantle density, $\boldsymbol{\rho_t}$: density of topography (usually taken to be $\mathbf{2670}\, kg \cdot m^{-3}$).

While we could choose any datum for levelling our survey results, it is conventional to choose a reference point for the modelling at the level at which the geoid intercepts the topographic plateau or mountain range and model a correction relative to the acceleration at that point. Globally, the depth to the crustal root at that point is about $\mathbf{30-33}$ kilometres. Knowing this, the density of the plateau, the height of the plateau, we can model the residual gravity deficiency across the highland and subtract the deficiency from our measurements. In order to calculate the correction, we first obtain the root depth as

$$\boldsymbol{d = dg + e(\frac{\rho_t}{\rho_m - \rho_t})}$$

over the land areas and

$$d = dg - dw(\frac{\rho_t - \rho_w}{\rho_m - \rho_t})$$

where ρ_w is normally taken to be that of sea-water, $\mathbf{1030}\,km \cdot m^{-3}$over the sea (sub-geoid) areas. Over regional areas small enough that we may disregard the sphericity of the Earth, this calculation, point-by-point is not difficult to model. Over distances where the curvature of Earth's surface begins to matter, In 1953, Heiskanen showed that the the gravitational effect of the root at any point of observation, O, along the profile on the geoid was

$$\Delta g_z = Gm \frac{(\frac{a^2}{2R} + d\cos\alpha)}{(a^2 + d^2 - 2ad\sin(\alpha/2))^{3/2}}$$

and d is the root depth at the point of contribution to the regional anomaly. (see Figure 16.).

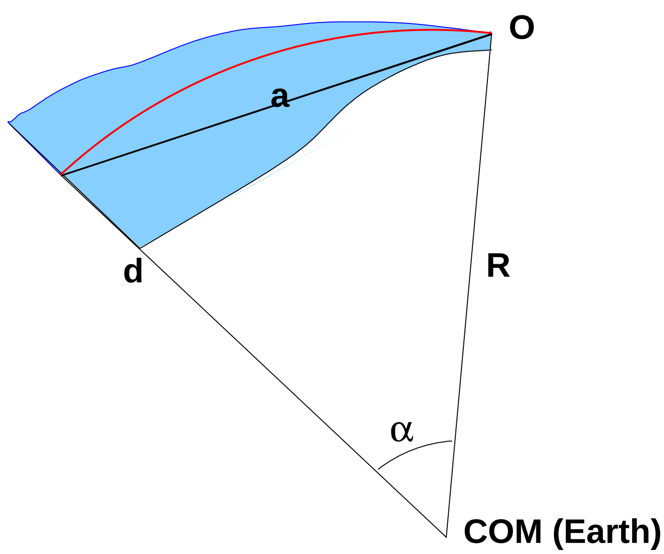


Figure 16: Regional corrections on a spherical Earth. From the point of observation, O, the contribution at angular distance, α, arc distance, a.

While the geoid is essentially mapped using satellite geodetic techniques, the gravitational acceleration field has normally been mapped by national geophysical ground surveys[19]. The latter work is laborious and so, with some lesser degree of sensitivity to near surface density anomalies and poorer resolution with respect to placement

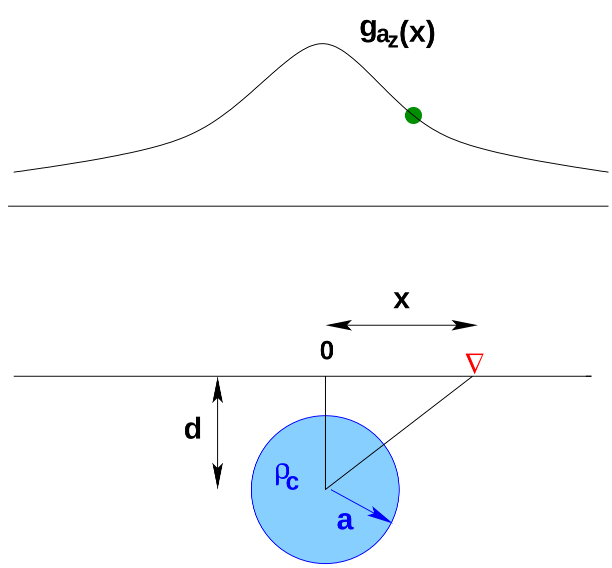
[19] Canada: Geoscience Data Repository

and extent, satellite techniques[20] are now being employed to measure the gravity field. Gravitational acceleration measurements obtained by satellite gradiometry are downward continued to the surface using Laplace's equation under the assumption that the intervening atmospheric mass is either uniform over areas or barometrically measured.

1.5.5 Some simple modelling of gravity anomalies:

While the principles of the process for the modelling of anomalies in surface gravitational accelerations have been essentially covered above, here, we shall look to a few useful and simple models.

- **A buried sphere:** The gravitational acceleration due to a volume with mass-density anomaly $\boldsymbol{\rho_c}$ with perfectly spherical distribution can be exactly replaced by a point mass model with that mass centred on the sphere's centre of mass. Suppose we were to have a spherical mass of density contrast $\boldsymbol{\rho_c}$ with the host rock, having a radius $\boldsymbol{a}$ and buried at depth $\boldsymbol{d}$. The mass contrast with the host rock would be $\boldsymbol{M_c = \frac{4}{3}\pi a^3}$. We calculate the vertical component of the gravitational anomaly caused by this body as measured from a flat surface. Note that if we were to measure the gravity anomaly with instruments across the ground surface above this mass, we would employ all of the corrections to the acceleration measurements as explained above before comparing our measurements with the theoretical anomaly.



[20] The major current and recent gravity missions are known by acronyms: **GRACE, GGOS, GOCE**

Figure 17: Calculating the gravity anomaly due to a buried sphere.

At the measurement point ▽ in Figure 17, the magnitude of the anomalous gravitational acceleration directed from the measurement point to the body's centre of mass is simply

$$g_a = \frac{GM_c}{d^2 + x^2}$$

and its vertical component, that which we would usually measure with a gravity meter, would be scaled by

$$\frac{d}{\sqrt{d^2 + x^2}}.$$

The downward-directed vertical component gravity anomaly at horizontal distance x from the centre of the spherical mass is then, simply,

$$g_{a_z}(x) = GM_c \frac{d}{(d^2 + x^2)^{\frac{3}{2}}};$$

its maximum value when $x = 0$ would be

$$g_{a_z}(x = 0) = GM_c \frac{1}{d^2}.$$

At some separation $x_{\frac{1}{2}}$, the *half-width at half-height* (point ●), the anomaly would be

$$g_{a_z}(x_{\frac{1}{2}}) = \frac{1}{2} g_{a_z}(x = 0).$$

A simple calculation obtains $d = 1.305 x_{\frac{1}{2}}$ and we have a simple method for determining the depth to the centre of mass from our measurement datum plane.

While the buried sphere is the most nearly trivial of models, it is an extremely useful one for describing the essential gravity anomaly due to buried compact bodies. We shall continue with a few other standard models.

- **A buried, infinitely long, horizontal cylinder:** Many anomalous masses are not constrained with small dimensional range in all three component directions. For a body that is constrained to small dimensions in the x and z direction but not in the y direction, we might approximate it by an infinite horizontal cylinder. We call this a *2-dimensional* model because 1 dimension, the y is left unconstrained.

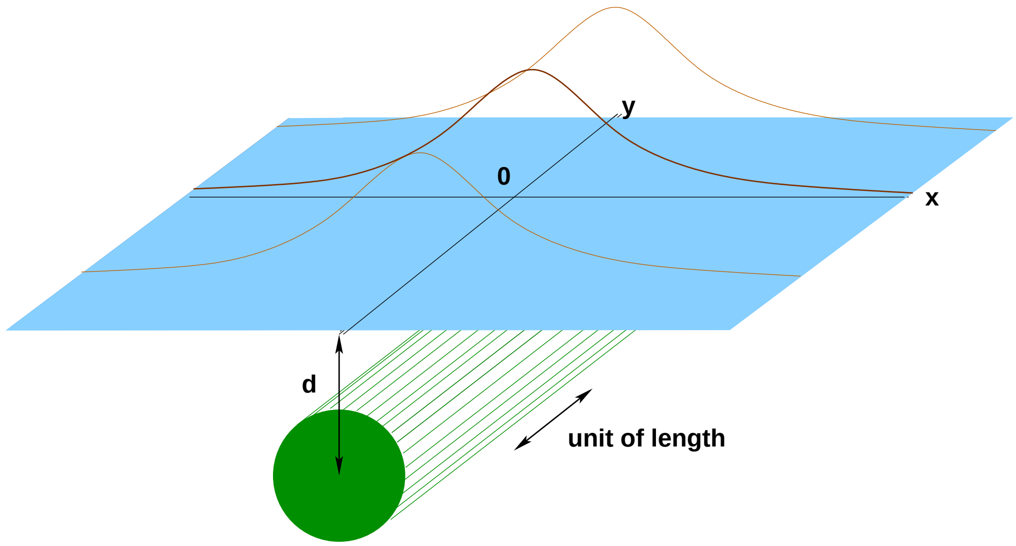


Figure 18: Calculating the gravity anomaly due to a buried horizontal cylinder.

We calculate, rather directly, the gravity acceleration toward the cylinder along a line parallel to the $\boldsymbol{y}$-axis. Again, we recognize that the centre of mass of the infinite cylinder, assuming its contrast density is constant, forms a line along the central axis of the cylinder. Let $\mathfrak{m}$ represent the mass per unit length along the axis of the cylinder. At distance $\boldsymbol{x}$ from the cylinder's axis, the gravitational acceleration due to an element of unit length separated by distance $\boldsymbol{y'}$ from the $\boldsymbol{y}$-origin contributes

$$d\vec{g}_a = -\frac{G\mathfrak{m}}{r^2}$$

where $r = \sqrt{d^2 + x^2 + y'^2}$ and with direction toward the element of mass. Then by integrating all the unit-length mass elements from $y' = -\infty$ to $y' = +\infty$, we obtain the acceleration anomaly directed toward the centre line of the cylinder at position $x, y = 0$ on the surface[21] :

$$\vec{g}_a(x, y) = G\mathfrak{m}\int_{-\infty}^{+\infty} \frac{1}{d^2 + x^2 + y'^2} dy' = \frac{2\pi G\mathfrak{m}}{\sqrt{d^2 + x^2}}.$$

[21]For a cylinder truncated at $y' = \pm Y$, this integral becomes

$$\vec{g}_a(x, y = 0) = \frac{2G\mathfrak{m}\tan^{-1}\left(\frac{Y}{\sqrt{d^2+x^2}}\right)}{\sqrt{d^2 + x^2}}.$$

At distance $\boldsymbol{x}$ from the origin, the vertical acceleration is simply scaled by $\boldsymbol{d}/\sqrt{\boldsymbol{d}^2 + \boldsymbol{x}^2}$ and

$$\boldsymbol{g_{a_z}} = \frac{\boldsymbol{2\pi G \mathfrak{m}}}{\boldsymbol{d}^2 + \boldsymbol{x}^2}.$$

The half-width at half-height is trivially calculated: $\boldsymbol{x_{\frac{1}{2}} = d}$.

Most introductory textbooks in geophysics and applied geophysics obtain many *standard* gravitational anomaly models. The principles have been shown for example. Still, these two models are among the most informative in interpreting gravity anomalies. There is no *deeper* compact (i.e. 3 constrained dimensions) structure that determines a *narrower* anomaly than that of the buried sphere. That means that if one does measure a compact anomaly, we can use the half-width at half-height to determine the deepest possible depth of its centre of mass. This is useful in a mineral prospect where one might be siting drilling to obtain geological samples. Whatever you are drilling for won't be deeper than $\boldsymbol{x_{\frac{1}{2}}}$. Similarly for anomalies that show very long strike, we might use the half-width at half-height obtained by survey on a profile normal to the strike to determine the maximum depth to the centre of mass of long bodies. One can, of course, engage in elaborate modelling for any imagined structure and adjust the structural model to fit, as closely as we may want, anomaly measurements. It should though be noted that this process can obtain an infinite number of very different structural-density models that can exactly fit any anomaly. That is, there is an inherent *ambiguity* in modelling. We can find the anomaly caused by any structural and density variations if we know the variations but we can't find the model that explains our anomaly measurements uniquely. This is a characteristic of all *geophysical inverse problems*. We can always determine the geophysical anomaly due to geophysical variations (*the forward problem)*; we can't uniquely determine the geophysical variations from geophysical measurements (*the inverse problem)*.

1.6 Geological interpretations of gravitational potential and acceleration surveys

Many geologists see geophysical technologies and the geophysicists who provide them as something of a *deputy* or *service* science for geology. Many geophysicists see their science as fundamental and almost independent of geological investigations. They, like I, have often chosen to focus their interests below the geological surface of the planet or to other planets of the solar system. The space programs are largely geophysical, but even then, much of what we can learn about other representative planets and moons in the solar system is geologically or geographically limited to the surface. Perhaps the only core, solid body geophysics for the planets and moons relates to their magnetic fields and internal mass distributions reflected in their rotation moments of inertia.

Geological interpretation of geophysical data probably requires more understanding of geological processes and conditions than understanding of geophysical science. Here, I shall retreat to my rather limited geological knowledge and experience... less than that of many of you in this class. I also retreat to a PowerPoint presentation of the lecture: **Interpreting Gravity**.